ANALYTICAL-NUMERICAL METHOD FOR SOLVING NONLINEAR DYNAMICAL SYSTEMS

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ABSTRACT

This paper investigates the analytical-numerical method for solving nonlinear dynamical systems. The governing equation of partial differential equation of order four was transformed to Ordinary differential equation using analytical method. The finite difference method was used to transform the approximate governing equation. It was shown from the graph of deflection against distance that the deflection increases as the value of distance increases and also shown from the graph of deflection against time that the deflection increases with increase in time. The result is in agreement with the existing results.

Keyword: Deflection, Dynamical Systems, Finite difference Method, Load, Nonlinear Equations, Systems

INTRODUCTION

The moving load is an unavoidable difficulty in structural dynamics. The dynamic behaviour of beams on elastic foundations subjected to moving loads or masses has been investigated by many researchers in engineering, especially in Railway Engineering, Adel *et al,* (2011). The modern trend towards higher speeds in the railways has further intensified the research in order to accurately predict the vibration behaviour of the railway track, Dehestain (2009). These studies mostly considered the Winkler elastic foundation model that consists of infinite closely-spaced linear springs subjected to a moving load, Fryba (1972).

The dynamic response of structures carrying moving masses is a problem of wide spread practical significance, Gbadeyan, *et al,* (1992). A lot of hard work has been done during the last 100 years relating with the dynamic response of railways bridges and highway bridges under the effect of moving loads, Ojih, *et al* (2014). Beam type structures are widely used in many branches of civil, mechanical and aerospace engineering, Gbadeyan, *et al* (1995). The dynamic effect of moving loads was not known until mid-nineteenth century. When the Stephenson's bridge across river Dee Chester in England in 1947 collapsed, it motivates the engineers for research of moving load problem, Lueschen, *et al*, (1996), Michaltsos, *et al*. (2001), Gbolagade, *et al,* (2003).

The simplest case of a moving load investigation is the case of a simple beam over which a concentrated load is moving, that is represented with a Fourth order *partial* differential equation, Jia-Jong, *et al,* (2000). This problem has significant effect in civil and mechanical engineering, Siddiqui, *et al,* (2003). The dynamic analysis of the vibrating beam is done by neglecting the disconnection of the moving mass from the beam during the motion and the result is given by considering mass moving at constant speed and in one direction, Muhamed, *et al,* (2007). Once the load departs from the beam, it begins to vibrate at in free vibration mode. Hence this process no longer comes within the scope of the experiment, Karghmovin, *et al,* (2004).

The problem of moving loads on structures was first considered in the early Nineteenth century when the traversing of bridges by locomotives was analysed, this has been followed by a considerable amount of research on this topic, PiotriKoziol, *et al,* (2012). The purpose of dynamic analysis is to know the structural behaviour under the influence of various loads and to get the necessary information for design such as deformation, moments and dynamic forces etc. Structural analysis is classified in to static and dynamic analysis. Static analysis deals with load which is time independent, Ugural (1981). But in dynamic analysis magnitude, direction and position of mass change with respect to time. Important dynamic loads for vibration analysis of bridge structure are vehicle motion and wave actions i.e. earthquake, stream flow and winds, Mohan (2012).

Lee (1998) studied extensively the dynamic responses of a beam acted upon by moving forces or moving masses, in connection with the design of railway tracks and bridges and machining processes, Foda, *et al,* (1998). The equation of motion in matrix form has been formulated for the dynamic response of a beam acted upon by a moving mass by using the Lagrangian approach, Umian (2009). Convergence of numerical results is found to be achieved with just a few terms for the assumed deflection function, Kargarnovin, *et al,* (2004) also analysed the dynamic response of infinite Timoshenko and Euler-Bernoulli beams on nonlinear viscoelastic foundations to harmonic moving loads, Gurgoze, (2001), Seong-Min (2004).

Mehri, *et al,* (2009) presented the linear dynamic response of uniform beams with different boundary conditions excited by a moving load, based on the Euler- Bernoulli beam theory, Kukla, (1997), Kukla, *et al,* (1994). Using a dynamic green function, effects of different boundary conditions, velocity of load and other parameters are assessed and some of the numerical results are compared with those given in the references, Kargarmovin, *et al,* (2005).

MATHEMATICAL FORMULATION

In this section, the dynamic response of a Bernoulli Beam on Winkler foundation under the action of moving partially distributed load is analysed and investigated.

The resulting vibrational behaviour of this system is described by the following partial differential equations, Gbadeyan, *et al,* (1992).

$$EIW_{xxxx}(x,t)+\frac{m}{L}W_{tt}(x,t)+KW(x,t)=f(x,t) \qquad (1.0)$$

where $f(x,t)$ is the applied moving mass defined as

$$f(x,t)=\left(\frac{1}{\epsilon}\right)\left\{-Mg-MW_{tt}(x,t)\right\}\left[H\left(x-\varepsilon+\frac{\epsilon}{2}\right)-H\left(x-\varepsilon-\frac{\epsilon}{2}\right)\right] \qquad (1.1)$$

NOMENCLATURE

Parameter	Description
L	Beam length
EI	Flexural rigidity of the beam
E	Modulus of Elasticity
W(x,t)	The lateral deflection of the beam measured upwards from its equilibrium when unloaded axial coordinate.
x	Axial coordinate
K	The coefficient of Winkler foundation (force per length squared)
m	The constant mass per unit length of the beam
M	The mass of the load
t	The time
g	The acceleration due to gravity
ϵ	Fixed length of the beam

Furthermore, the total derivative $W_{tt}(x,t)$ which appears in equation (3.1) is defined as

$$W_{tt}(x,t)=W_{tt}(x,t)+2VW_{xt}(x,t)+V^2W_{xx}(x,t) \quad (1.2)$$

Where V is the constant velocity of the moving mass which is defined as

$$\varepsilon = Vt + \frac{\epsilon}{2} \quad (1.3)$$

$H(x)$ is the Heaviside unit function usually defined as

$$H(x)=\begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases} \quad (1.4)$$

BOUNDARY CONDITIONS

The pertinent boundary conditions for the problem under consideration can be any of the following classical boundary conditions.

$$\left.\begin{aligned} W(x,t)=W_t(x,t)=0 \quad & \text{at } x=0 \text{ or } x=l \\ W(x,t)=W_{tt}(x,t)=0 \quad & \text{at } x=0 \text{ or } x=l \\ W_{xx}(x,t)=W_{xxx}(x,t)=0 \quad & \text{at } x=0 \text{ or } x=l \\ W_x(x,t)=W_{xxx}(x,t)=0 \quad & \text{at } x=0 \text{ or } x=l \end{aligned}\right\} \quad (1.5)$$

Finally, the initial conditions are:

$$\left.\begin{aligned} W(x,0)=0 \\ W_t(x,0)=0 \end{aligned}\right\} \quad (1.6)$$

SOLUTION OF THE PROBLEM

In this section, we proceed to solve the above initial boundary – value problem described by equation (1.0), (1,1) and (1.6)

To this effect, we assume that the unknown initial deflection, $W(x,t)$ of the beam resting on Winkler foundation can be expressed as:

$$W(x,t)=\sum_{j=1}^{\infty} T_j(t)X_j(x) \quad (1.7)$$

Where $T_j(t)$ are unknown functions of time t and $X_j(x)$ are the normalized deflection curves for the j^{th} mode of the vibrating non-prismatic beam.

After taking the derivatives of equation (1.7), equation (1.0) becomes

$$EI\sum_{j=1}^{\infty} T_j(t)X_j^{(iv)}(x)+\frac{m}{L}\sum_{j=1}^{\infty} T_j(t)X_j(x)+K\sum_{j=1}^{\infty} T_j(t)X_j(x)=f(x,t) \quad (1.8)$$

At this juncture, it is remarked that the applied force can also be expressed as a series solution to equation (3.7)the we have

$$f(x,t)=\sum_{j=1}^{\infty} T_{fj}(t)X_j(x) \tag{1.9}$$

Where T_{fj} are unknown functions of time different from those T_j. Equations (1.8) and (1.9) yield

$$EI\sum_{j=1}^{\infty} T_j(t)X_j^{(iv)}(x)+\frac{m}{L}\sum_{j=1}^{\infty} T_j(t)X_j(x)+K\sum_{j=1}^{\infty} T_j(t)X_j(x)$$
$$=\sum_{j=1}^{\infty} T_{fj}(t)X_j(x) \tag{2.0}$$

It is noted that equation (2.0) has two sets of unknowns viz: the T_j's and the T_{fj}'s. this naturally makes equation (2.0) highly coupled. To reduce his high degree of coupleness, we would have to determine one of the these sets of unknowns. We remark, however, that we find it convenient to determine the T_{fj}'s. To this end, we first notice that equations (1.1) and (1.9) yield

$$\sum_{j=1}^{\infty} T_{fj}(t)X_j(x)=\left(\frac{1}{\epsilon}\right)\left\{-Mg\left[H\left(x-\varepsilon+\frac{\epsilon}{2}\right)-H\left(x-\varepsilon-\frac{\epsilon}{2}\right)\right]\right\}$$
$$\left[H\left(x-\varepsilon+\frac{\epsilon}{2}\right)-H\left(x-\varepsilon-\frac{\epsilon}{2}\right)\right] \tag{2.1}$$

Next, multiply equation (2.1) by the unknown normalized deflection function $X_i(x)$ and then integrate the resulting equation over the length of the beam to obtain

$$T_{fj}=Mg\left[X_i(\varepsilon)+\frac{\epsilon^2}{24}X_j''(\varepsilon)\right]$$
$$-M\sum_{j=1}^{\infty} T_j(t)\left\{X_j(\varepsilon)X_i(\varepsilon)+\frac{\epsilon^2}{24}\left[\begin{matrix}X_j''(\varepsilon)X_i(\varepsilon)+2X_j'(\varepsilon)X_i'(\varepsilon)\\+X_j(\varepsilon)X_i''(\varepsilon)\end{matrix}\right]\right\}$$
$$-2MV\sum_{j=1}^{\infty} T_j(t)\left\{X_j'(\varepsilon)X_i(\varepsilon)+\frac{\epsilon^2}{24}\left[\begin{matrix}X_j'''(\varepsilon)X_i(\varepsilon)+2X_j''(\varepsilon)X_i'(\varepsilon)\\+X_j'(\varepsilon)X_i''(\varepsilon)\end{matrix}\right]\right\}$$
$$-MV^2\sum_{j=1}^{\infty} T_j(t)\left\{X_j''(\varepsilon)X_i(\varepsilon)+\frac{\epsilon^2}{24}\left[\begin{matrix}X_j^{(iv)}(\varepsilon)X_i(\varepsilon)+2X_j'''(\varepsilon)X_i'(\varepsilon)\\+X_j''(\varepsilon)X_i''(\varepsilon)\end{matrix}\right]\right\} \tag{2.2}$$

Next substituting equation (2.2) into equation (2.0), the approximate governing equation is found to be

$$EI\sum_{j=1}^{\infty} T_j(t)X_j^{(iv)}(x)+\frac{m}{L}\sum_{j=1}^{\infty} T_j(t)X_j(x)+K\sum_{j=1}^{\infty} T_j(t)X_j(x)$$
$$=\sum_{j=1}^{\infty} X_j$$
$$\left(-Mg\left[X_i(\varepsilon)+\frac{\epsilon^2}{24}X_j''(\varepsilon)\right]-M\sum_{j=1}^{\infty} T_j(t)\left\{X_j(\varepsilon)X_i(\varepsilon)+\frac{\epsilon^2}{24}\left[\begin{matrix}X_j''(\varepsilon)X_i(\varepsilon)+2X_j'(\varepsilon)X_i'(\varepsilon)\\+X_j(\varepsilon)X_i''(\varepsilon)\end{matrix}\right]\right\}-2MV\sum_{j=1}^{\infty} T_j(\right. \tag{2.3}$$

To simplify equation (2.3), we noted that for free vibration of an Euler-Bernoulli beam, we have

$$X_j^{(iv)}(x)=\beta_j^4 X_j(x)=0 \tag{2.4}$$

$$\text{where} \qquad \beta_j^4=\frac{mP_j^2}{EI} \tag{2.5}$$

and P_j^2 is the square of the j[th] natural frequency of the beam.

For arbitrary $X_j(x)$, we have

$$T_j(t)+\left(P_j^2+\frac{K}{m}\right)T_j(t)$$

$$=\frac{1}{m}$$

$$\left(-Mg\left[X_i(\varepsilon)+\frac{\epsilon^2}{24}X_j''(\varepsilon)\right]-M\sum_{j=1}^{\infty}T_j(t)\left\{X_j(\varepsilon)X_i(\varepsilon)+\frac{\epsilon^2}{24}\left[\begin{matrix}X_j''(\varepsilon)X_i(\varepsilon)+2X_j'(\varepsilon)X_i'(\varepsilon)\\+X_j(\varepsilon)X_i''(\varepsilon)\end{matrix}\right]\right\}-2MV\sum_{j=1}^{\infty}T_j($$

(2.6)

Equation (2.6) is the desired set of coupled second order differential equations. By solving these equations in (2.6) for $T_j(t)$'s and substituting the resulting expression into equation (1.7), the desired solution for the vibration of the beam under different boundary conditions and with any number of modal shapes can be determined.

SIMPLY-SUPPORTED BEAM

To solve the above coupled equation (2.6), we need to know the exact form of the normalized deflection X_j. As a matter of fact, there is exists various turns of X_i depending on the vibrating configurations of the beam. In other words, the solution of equation (2.6) depends on the associated boundary conditions as the exact form of X_i depends on the type of boundary conditions under consideration.

Hence, as an illustrative example, we consider a beam which is simply supported that is a beam whose boundary conditions are given as

$$W(x,t)=0=W_{xx}(x,t)=0 \quad \text{at } x=0 \text{ and } x=L \qquad (2.7)$$

It is well known that for a simply supported beam

$$X_j(x)=\sqrt{\frac{2}{L}}\sin\frac{j\pi x}{L}, \quad j=1,2,3,\ldots \qquad (2.8)$$

Direct substituting of (2.8) into (2.6) will yield the desired governing equation which is, however an approximate one. It is remarked, that for the configuration under discussion an exact differential governing equation can be derived by going through arguments similar to those used in obtaining equation (2.6).

$$\sum_{j=1}^{\infty}T_{fj}\int_0^L\frac{2}{L}\sin\frac{j\pi x}{L}\sin\frac{i\pi x}{L}dx$$

$$=-\frac{Mg}{\epsilon}\int_0^L\sqrt{\frac{2}{L}}\sin\frac{i\pi x}{L}\left[H\left(x-\varepsilon+\frac{\epsilon}{2}\right)-H\left(x-\varepsilon-\frac{\epsilon}{2}\right)\right]dx$$

$$-\frac{M}{\epsilon}\{\sum_{j=1}^{\infty}T_j(t)\int_0^L\sin\frac{j\pi x}{L}\sin\frac{i\pi x}{L}\left[H\left(x-\varepsilon+\frac{\epsilon}{2}\right)-H\left(x-\varepsilon-\frac{\epsilon}{2}\right)\right]dx$$

$$+2V\sum_{j=1}^{\infty}T_j(t)\int_0^L\left(\frac{L}{j\pi}\right)\left(\frac{2}{L}\right)\cos\frac{j\pi x}{L}\sin\frac{i\pi x}{L}\left[H\left(x-\varepsilon+\frac{\epsilon}{2}\right)-H\left(x-\varepsilon-\frac{\epsilon}{2}\right)\right]dx$$

$$+V^2\sum_{j=1}^{\infty}T_j(t)\int_0^L\frac{2}{L}\left(\frac{-L^2}{i^2\pi^2}\right)\sin\frac{j\pi x}{L}\sin\frac{i\pi x}{L}\left[H\left(x-\varepsilon+\frac{\epsilon}{2}\right)-H\left(x-\varepsilon-\frac{\epsilon}{2}\right)\right]dx$$

(2.9)

After integrating equation (2.9) for arbitrary $X_j(x)$, we obtain

$$T_j(t)+\left(P_j^2+\frac{k}{m}\right)T_j(t)=\frac{1}{m}[-\frac{Mg}{j\pi\epsilon}\sqrt{8L}\sin\left(\frac{j\pi\varepsilon}{L}\right)\sin\left(\frac{i\pi\epsilon}{L}\right)\sum_{j=1}^{\infty}T_{fj}$$

$$-\frac{2M}{\epsilon\pi}\sum_{j=1}^{\infty} T_j(t)\frac{1}{(i-j)}\left\{\sin\left(\frac{j\pi\varepsilon}{2L}\right)\cos\frac{\pi\epsilon}{L}(i-j)\right\}+\frac{2M}{\epsilon\pi}\sum_{j=1}^{\infty} T_j(t)\frac{1}{(i+j)}\left\{\cos\frac{\pi\epsilon}{L}(i+j)\sin\frac{\pi\epsilon}{2L}(i+j)\right\}$$
$$+\frac{2MV}{\epsilon}\sum_{j=1}^{\infty} T_j(t)\sqrt{\frac{2}{L}}i\left\{\left[\sin\frac{\pi\epsilon}{L}(i+j)\sin\frac{\pi\epsilon}{2L}(i+j)\right]+\left[\sin\frac{\pi\epsilon}{L}(i-j)\sin\frac{\pi\epsilon}{2L}\frac{(i-j)}{(i-j)}\right]\right\}$$
$$+\frac{MV^2}{\epsilon}\left(\frac{i\pi}{L}\right)\sum_{j=1}^{\infty} T_j(t)\sqrt{\frac{2}{L}}\frac{1}{(i+j)}\left\{\left[\sin\frac{\pi\epsilon}{L}(i+j)\cos\frac{\pi\epsilon}{2L}(i+j)\right]+\left[\cos\frac{\pi\epsilon}{L}(i+j)\sin\frac{\pi\epsilon}{2L}(i+j)\right]\right\} \quad j=1,2,3,\ldots \quad i\neq j \qquad (3.0)$$

We now use the finite difference method in order to solve the above equation numerically, we made use of approximate central difference method, we obtain.

$$\left\{m+\frac{2M}{\epsilon\pi}\left[\frac{1}{(i-j)}\cos\frac{\pi\epsilon}{L}(i-j)\sin\frac{\pi\epsilon}{2L}(i-j)-\frac{1}{(i-j)}\cos\frac{\pi\epsilon}{L}(i+j)\sin\frac{\pi\epsilon}{2L}(i+j)\right]+\frac{hMVi}{\epsilon}\sqrt{\frac{2}{L}}\left[\frac{1}{(i+j)}\sin\frac{\pi\epsilon}{L}(i+j)s\right.\right.$$

$$T_{j+1}$$

$+$

$$\left\{-2M-\frac{4M}{\epsilon\pi}\left[\frac{1}{(i-j)}\cos\frac{\pi\epsilon}{L}(i-j)\sin\frac{\pi\epsilon}{2L}(i-j)+\frac{1}{(i-j)}\cos\frac{\pi\epsilon}{L}(i+j)\sin\frac{\pi\epsilon}{2L}(i+j)\right]+K+mh^2P_j^2+\frac{h^2MV^2i\pi}{\epsilon L}\sqrt{\frac{2}{L}}\left[(\right.\right.$$

$$T_j$$

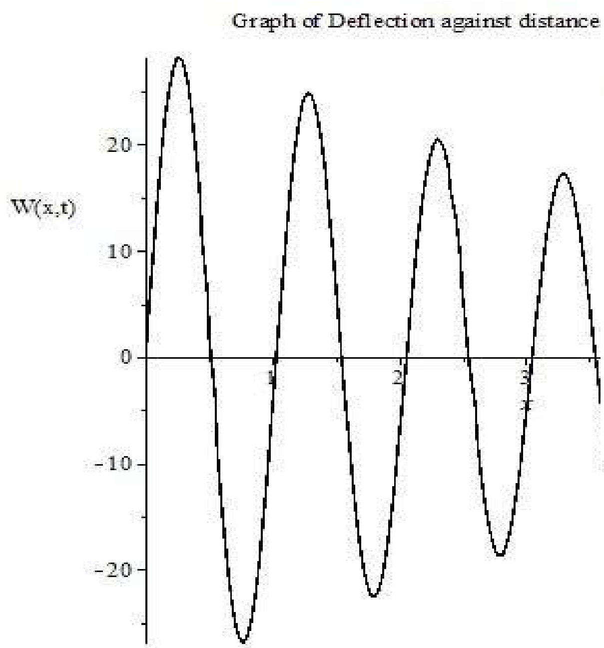
$+$

$$\left\{M+\frac{2M}{\epsilon\pi}\left[\frac{1}{(i-j)}\cos\frac{\pi\epsilon}{L}(i-j)\sin\frac{\pi\epsilon}{2L}(i-j)-\frac{1}{(i+j)}\cos\frac{\pi\epsilon}{L}(i+j)\sin\frac{\pi\epsilon}{2L}(i+j)\right]+\frac{hmVi}{\epsilon}\sqrt{\frac{2}{L}}\left[\frac{1}{(i+j)}\sin\frac{\pi\epsilon}{L}(i+j)c\right.\right.$$
$$T_{j-1}=-\frac{h^2Mg}{i\pi\epsilon}\sqrt{8L}\sin\left(\frac{j\pi\varepsilon}{L}\right)\sin\left(\frac{i\pi\epsilon}{L}\right) \qquad (3.1)$$

NUMERICAL RESULTS

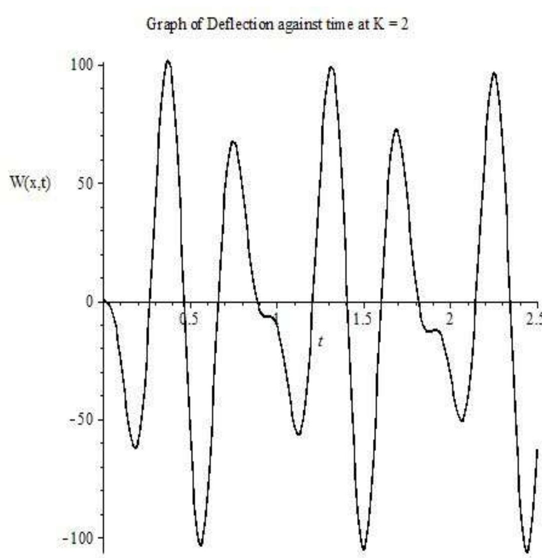
The result obtained in equation (3.1) for nonlinear dynamical systems subjected to partially distributed load is discussed in this chapter using analytical-numerical method. Which made use of approximate finite difference method and MATLAB was used for the values of the variable used and the following graphs were plotted as shown in figures 1.1- 1.8.

Fig 1.1



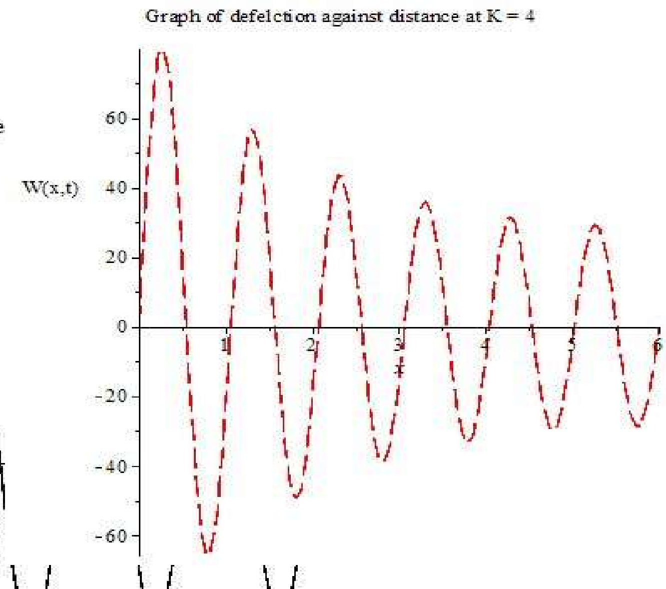
It is found that as the value of deflection increases and the value of distance increases there is a decrease in velocity deflection at K=2.

Fig 1.2



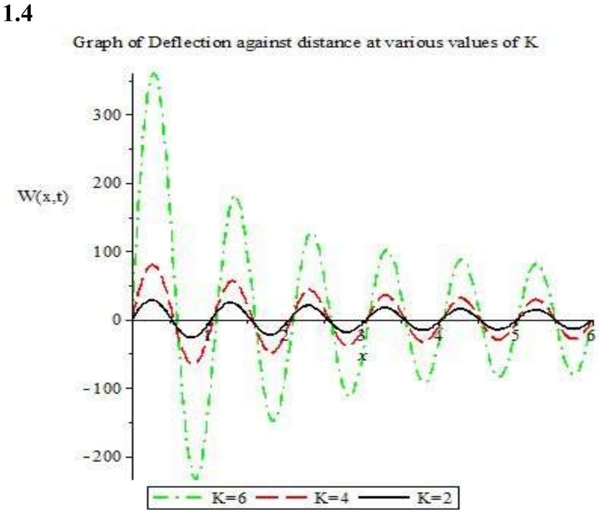
It is found that at a constant value of time (t) the deflection increases and also decreases at k=2.

Fig 1.3



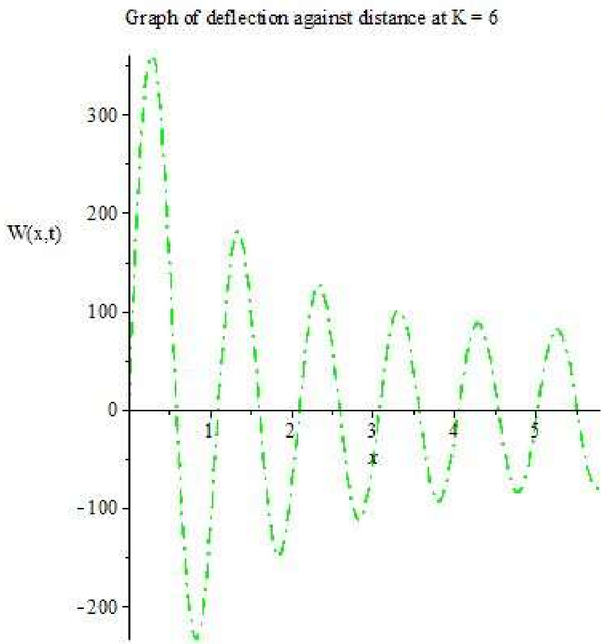
It is found that as the value of deflection increases and the value of distance increases there is a decrease in velocity deflection at K=4.

Fig 1.4

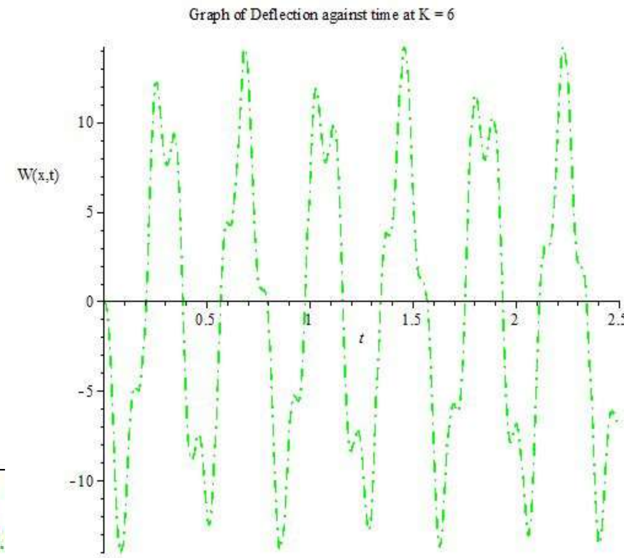


Deflection against distance at various values of K, which shows that at a constant value of velocity deflection increases as the value of K increases. This was in agreement with the existing result.

Fig 1.5

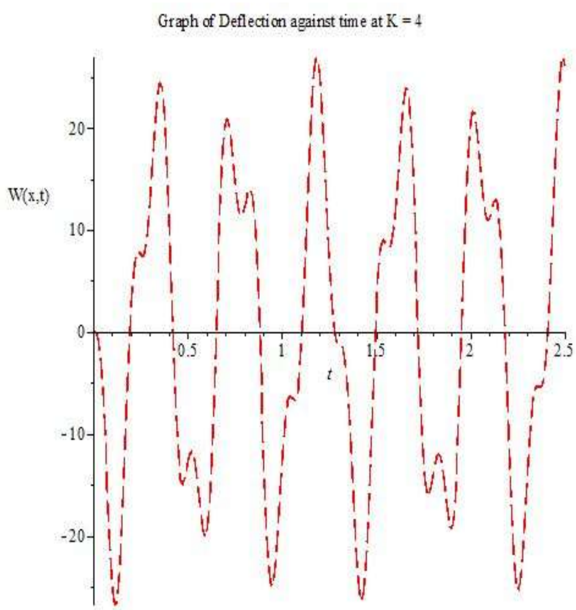


It is found that deflection decreases against distance at a constant value of x at K=2.

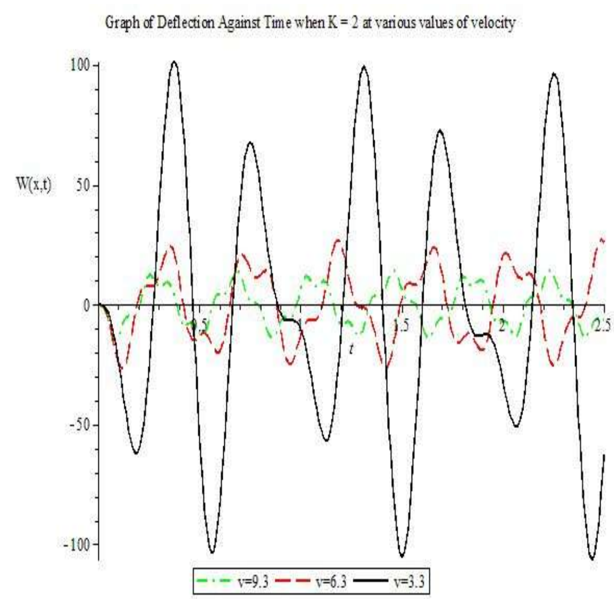


1.7 Deflection against time, it was found that the deflection decreases first and later increases at different time (t) which was the same result we are given in Fig 1.6.

Fig 1.6



Deflection against time, it was found that the deflection decreases first and later increases at different time (t) when K=4



1.8 Deflection against time When K=2 at various values of velocity V= 9.3, V=6.3 and V=3.3 at constant value of K deflection reduces as velocity increases.

NUMERICAL DISCUSSION

The dynamic response of nonlinear dynamical systems were considered. Also the systems were subjected to partially distributed load is observed for various values of deflections against time, at constant value of K deflection reduces as velocity increases and also for deflection against distance, at constant value of velocity deflection increases as the value of K increases.The results obtained were compared with the existing result.

REFERENCES

Adel A. Al-Azzawi (2011): Analysis of Timoshenko Beam resting on a nonlinear compressional and frictional Winkler foundation: ARPN, Journal of Engineering and Applied Sciences. Asian Publishing Network (ARPN). 6. No 11. 1 – 14.

Dehestani, M. Mofid, A. Vafai (2009): Investigation of critical influential speed for moving mass problems on beams,Applied Mathematical Modeling 33,3885-3895

Fryba, L. (1972): "Vibration of Solids and Structures under Moving Loads",Noordhoff International, Groningen.

Gbadeyan J.A. and Oni S.T. (1992): Dynamic Response of Moving Concentrated Masses of Elastic Plates on a non-Winkler Elastic Foundation. Journal of Sound and Vibration 154(2): 343 - 358.

Gbadeyan J.A. & Oni S.T. (1995): Dynamics behaviour of Beams and rectangular plates under moving loads. Journal of Sound and Vibration. 182 (5,. 677 – 695.

Lueschen, G.G.G, Bergman & Macfarland, D.M. (1996): Green functions for uniform Timoshenko Beams, Journal of sound and vibration 194, 93-102.

Michaltsos, G.T. & Kounadis, A.N. (2001): The Effects of Centripetal and Coriolis Forces on the Dynamic Response of Light Bridges Under Moving Loads,Journal of Sound and Vibration 247,261-277.

Gbolagade, A.W ,Gbadeyan, J.A., S.O.S. Olabamiji, O. Otolorin (2003): Response of Elastically Connected Beams subjected to moving forces .Nigeria Journal of Mathematics and Applications, 16: 67-79.

Jia-Jang Wu, A.R. Whittaker, M.P. Cartmell (2000): The use of finite element techniques for calculating the dynamic response of structures to moving loads,Computers& Structures 78, 789-799.

Kargarnovin M. H. &Younesian, D. (2005): Response of beams on nonlinear viscoelastic foundations to harmonic moving loads, *Computers & Structures,* 83 1865-1877.

Kargarnovin, M. H. &Younesian, D. (2004): Dynamics of Timoshenko beams on Pasternak foundation under moving load, *Mechanics Research Communications*, 31, 713-723.

Lee, H. P. (1998) : Dynamic response of a Timoshenko beam on a Winkler foundation subjected to a moving mass, *Applied Acoustics*, 55 (3) 203-215.

M. Gurgoze, H. Erol (2001); Determination of the frequency response function of a cantilevered beam simply of supported in-span, Journal of sound and vibration 247, 372 – 378.

M.A Foda, Z. Abduljabbar (1998); A dynamic Green function formulation for the response of a beam structure to a moving mass, Journal of sound and vibration 210, 295 – 306.

Mehri, B. A Davar, &Rahmani O. (2009): Dynamic Green Function Solution of Beams Under a Moving Load with Different Boundary Conditions. ScientiaIranica 16, 273-279.

Michaltsos, G.T. (2001): Journal of Sound and Vibration 258,359-372. Dynamic behavior of a single-span beam Subjected to loads moving with variable speeds.

Mohan CharanSethi (2012) ; Dynamic response under moving mass, a thesis submitted in partial fulfillment of the requirement for the degree of bachelor of technology in mechanical Engineering, Department of Mechanical engineering, National institute of technology, Rourkelas.

MuhammedUncuer, BozidarMarinkovic and HurKoser (2007): Simulation of damped – free and damped-damped microbeams dynamics for nonlinear mechanical switch application: Except from the proceeding of the COMSOL, Conference 2007.

Ojih, P.B., Ibiejugba M.A. and Adejo B.O. (2014): Dynamic Response under moving concentrated loads of uniform Rayleigh beam resting on Pasternak foundation. International Journal of Scientific and Engineering Research: Volume 5, Issue 6: 1 – 21.

PiotriKoziol and ZdzislawHryniewicz (2012): Dynamic Response of a beam resting on a nonlinear foundation to a moving load: Coiflet - based solution shock and vibration 19:995-1007.

S. Kukla, (1997); Application of Green function in frequency analysis of Timoshenko beams with oscillators, Journal of Sound and vibration 205, 355 – 363.

S. Kukla, B. Posiadala (1994); Free vibration of beams with elasticity mounted masses, Journal of sound and vibration 175, 557 – 564.

Seong-Min Kim, (2004): Vibration and Stability of axial loaded beams on elastic foundation under moving harmonic loads, *Engineering Structures,* 26, 5-105.

Siddiqui, S.A.Q., Golnaraghi, M.F. &Heppler (2003): Large free vibrations of beam carrying a moving mass. International journal of Nonlinear Mechanics 38, 1481-1493.

Ugural A.C. (1981): Stress in plates and shells. New York, McGraw – Hill.

Umian Essendemir (2009): Derivation of Equations for Flexure and Shear Deflections of Simply Supported Beams. Vol 6. 187 – 193.