ANALYTICAL SOLUTION OF THE LINEAR AND NONLINEAR KLEIN-GORDON EQUATIONS

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ABSTRACT

In this study, three powerful methods, the VIM, NIM and ADM were applied to find the solution of the linear and nonlinear Klein-Gordon equations. To illustrate the ability and reliability of the methods some examples were provided. In this study, we compare numerical results with the exact solution. The results show that the Variational Iteration Method, Adomian Decomposition Method and New Iterative Method are powerful and effective tools in solving the Klein-Gordon equations and can be used to solve other linear and nonlinear equations, ordinary and partial equations.

Keywords: Klein-Gordon equation, Variational iteration Method, Lagrange multiplier, New Iterative Method, Relativistic Wave Equation.

Introduction

Klein-Gordon equation plays an important role in mathematical physics, it appears in quantum field theory, dispersive wave-phenomena, plasma physics, nonlinear optics and applied and physical sciences.

We consider the Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + Nu(x,t) = f(x,t) \tag{1}$$

Subject to the initial conditions

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x) \tag{2}$$

where u is a function of x and t, $Nu(x,t)$ is a nonlinear function and $f(x,t)$ is a known or given analytical function (Al-luhaibi 2015). Because of the importance of Klein-Gordon equation in quantum mechanics, several techniques have been developed in order to compute the solution of these equations, such as Sumudu Decomposition Method, Perturbation Method Homotropy Perturbation Transform Method, New Perturbation Iteration Transform Method (Cheniguel et.al, 2011).

In this study, three (3) of these methods were applied to solve the Klein-Gordon equation, these methods are Variational Iteration Method (VIM), New Iterative Method (NIM) and Adomian Decomposition Method (ADM) with its modification (Gupta Praveen, 2012). These methods have been widely used in solving different types of differential equations in Physics, Engineering and Modelling (Hemeda , 2013). The VIM has been thoroughly used by mathematicians to handle a wide variety of scientific and engineering applications: linear and nonlinear, and homogeneous and inhomogeneous the Schrodinger equation, the Convection-diffusion equation, homogeneous and inhomogeneous partial differential equation, and many more, the NIM has been extensively used by many researchers for the treatment of linear and nonlinear ordinary and partial differential equations of integer and fractional order such as fractional physical differential equations, higher order KDV, Fractional Gas Dynamics and Coupled Burger's Equations, Nonlinear Abel type integral equation, linear and many more while the ADM has been used successfully to solve problems such as the nonlinear Sturm-Liouville problem, the fourth-order integra-differential equations, the heat equation, and many (Hosseinzadeh . 2017). They have been proved to be powerful, reliable, which can effectively easily and accurately solve higher order linear and nonlinear differential problems with rapid convergence with number of iterations (Ishak Hashim 2006).

METHODOLOGY

VARIATIONAL ITERATION METHOD (VIM)

Consider the following differential equation:

$$Lu + Nu = g(t) \tag{3}$$

Where, L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term.

The variational iteration method presents a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\varepsilon)\big(Lu_n(\varepsilon) + N\tilde{u}_n(\varepsilon) - g(\varepsilon)\big)d\varepsilon \tag{4}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$ (Manoj Kumar, et.al 2016).

It is obvious now that the main steps of the variational iteration method require first the determination of the Lagrange multiplier $\lambda(\varepsilon)$ that will be identified optimally (Khalid, et.al. 2016). Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\varepsilon)$. In other words, carrying out the integration as follows can yield:

$$\int \lambda(\varepsilon)\, u'_n(\varepsilon)d\varepsilon = \lambda(\varepsilon)u_n(\varepsilon) - \int \lambda'(\varepsilon)\, u_n(\varepsilon)d\varepsilon,$$
$$\int \lambda(\varepsilon)\, u''_n(\varepsilon)d\varepsilon = \lambda(\varepsilon)'u_n(\varepsilon) - \lambda'(\varepsilon)u_n(\varepsilon) + \int \lambda''(\varepsilon)\, u_n(\varepsilon)d\varepsilon \tag{5}$$

Having determined the Lagrange multiplier $\lambda(\varepsilon)$, the successive approximations u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using any selective function u_0 (Majeed . et.al 2017).
However, for fast convergence, the function $u_0(x,t)$ should be selected by using the initial conditions as follows:

$$u_0(x,t) = u(x,0) \quad \text{for first order}$$
$$u_0(x,t) = u(x,0) + tu_t(x,0) \quad \text{for second order} \tag{6}$$

Consequently, the solution

$$u = \lim_{n\to\infty} u_n \tag{7}$$

In other words, equation (5) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations (Olayiwola 2015).

NEW ITERATIVE METHOD

To illustrate the idea of the NIM, we consider the following general functional equation:

$$u = f + N(u) \tag{8}$$

where N is a nonlinear operator and f is a given function. We can find the solution of equation having the series form

$$u = \sum_{i=0}^{\infty} u_i \tag{9}$$

The nonlinear operator N can be decomposed as:

$$N\Big(\sum_{i=0}^{\infty} u_i\Big) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \tag{10}$$

Substituting equations (9) and (10) into equation (8) gives

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \tag{11}$$

We define the recurrence relation of equation in the following way:

$$u_0 = f$$
$$u_1 = N(u_0)$$
$$u_2 = N(u_0 + u_1) - N(u_0) \tag{12}$$
$$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1)$$
$$u_{n+1} = N(u_0 + u_1 + \ldots + u_n) - N(u_0 + u_1 + \ldots + u_{n-1}); \quad \text{n =1, 2, 3}$$

Then

$$u_1 + \cdots + u_{m+1} = N(u_0 + u_1 + \cdots + u_m); \quad \text{m = 1,2, 3}$$

and

$$\sum_{i=0}^{\infty} u_i = f + N(\sum_{j=0}^{\infty} u_j) \tag{13}$$

The m-term approximate solution of (8) is given by $u = u_0 + u_1 + u_2 \ldots + u_{m-1}$

ADOMIAN DECOMPOSITION METHOD

To give a clear overview of Adomian decomposition method,
Considering the following equation:

$$Lu + Ru = g \tag{14}$$

where L is, mostly, the lower order derivative which is assumed to be invertible, R is a linear differential operator, and g is a source term.

Applying the inverse operator (L^{-1}) to both sides of (14) and using the initial condition to obtain

$$u = f - L^{-1}Ru \tag{15}$$

where the function f represents the terms arising from integrating the source term g and noting the prescribed conditions.

The Adomian Decomposition Method assumes that the unknown function u can be expressed by an infinite series of the form

$$u(x,y) = \sum_{n=0}^{\infty} (u_n(\mathrm{x}, \mathrm{y})) \tag{16}$$

or equivalently,

$$u = u_0 + u_1 + u_2 + \ldots\ldots \tag{17}$$

where the components $u_0, u_1, u_2, \cdots$ are usually recurrently determined.
Substituting equation (16) into equation (15) yields

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R(\sum_{n=0}^{\infty} u_n)\right) \tag{18}$$

For simplicity, Equation (18) can be re-written as

$$u_0 + u_1 + u_2 + \cdots = \sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R(\sum_{n=0}^{\infty} u_0 + u_1 + u_2 + \cdots)\right) \tag{19}$$

Accordingly, the formal recursive relation is defined by

$$u_0 = f$$
$$u_{k+1} = -L^{-1}\big(R(u_k)\big) \quad , \quad k \geq 0 \tag{20}$$

or equivalently,

$$u_0 = f$$
$$u_1 = -L^{-1}\big(R(u_0)\big)$$
$$u_2 = -L^{-1}\big(R(u_1)\big)$$

$$u_3 = -L^{-1}\left(R(u_2)\right) \tag{21}$$

The approximate solution is given by $u = u_0 + u_1 + u_2 \dots + u_{m-1}$

MODIFIED ADOMIAN DECOMPOSITION METHOD

Modified Adomian decomposition method developed by. The modified decomposition method will further accelerate the convergence of the series solution. It is to be noted that in this study, the modified decomposition method will be applied to linear inhomogeneous and nonlinear Klein-Gordon equations (Sennur et.al 2007).

The decomposition method admits the use of the recursive relation,

$$u_0 = f,$$
$$u_{k+1} = -L^{-1}(Ru_k), \qquad k \geq 0 \tag{22}$$

the components u_n, $n \geq 0$ is obtained.

The modified decomposition method introduces a slight variation to the recursive relation (22) that will lead to the determination of the components of u in a faster and easier way.

For specific cases, the function f can be set as the sum of two partial functions, namely f_1 and f_2.

In other words, we have

$$f = f_1 + f_2. \tag{23}$$

In other words, the modified recursive relation can be identified by

$$u_0 = f_1$$
$$u_1 = f_2 - L^{-1}\left(R(u_0)\right) \tag{24}$$
$$u_{k+1} = -L^{-1}\left(R(u_k)\right)$$

The success of this modification depends only on the choice of f_1 and f_2, and this can be made through trials. Second, if f consists of one term only, the standard decomposition method should be employed in this case.

The approximate solution is given by $u = u_0 + u_1 + u_2 \dots + u_{m-1}$

NUMERICAL EXAMPLES

In this section, we use the three methods in solving linear and nonlinear Klein-Gordon equations.

Linear Klein-Gordon Equation

Example 1

Consider the Klein-Gordon equation

$$u_{tt} - u_{xx} - u = 0$$

With initial conditions

$$u(x,0) = Sin(x) + 1, \; u_t(x,0) = 0 \tag{25}$$

and the exact solution is $\quad u(x,t) = sin(x) + cosh(t)$

Following the procedures for the three methods after 3 iterations, the solution becomes

$$u(x,t) = sin(x) + cosh(t) \tag{26}$$

Which is the exact solution.

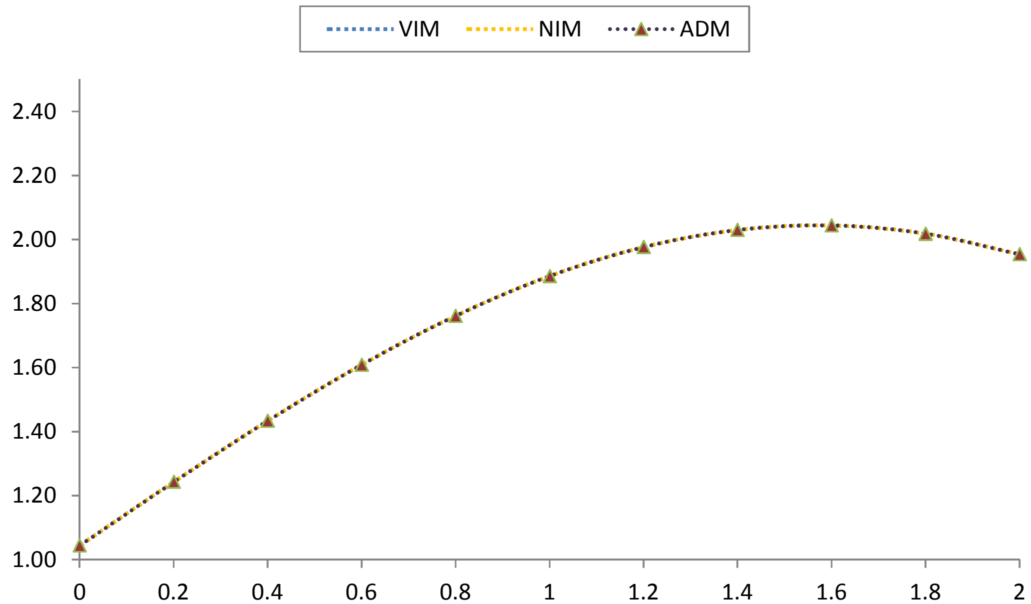

Figure 1: Comparison of the three methods for example 2 after 2nd iteration

Example 2

Consider the following Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0$$

With the initial conditions

$$u(x,0)=0,\ u_t(x,0) = x \tag{27}$$

and the exact solution is

$$u(x,t) = x\,\mathrm{Sin}\,t$$

Following the procedures in section 2 and by substituting the obtained coefficient in the equation, the solution for the three methods become

$$u(x,t) = x\,\mathrm{Sin}\,t \tag{28}$$

Which is the exact solution.

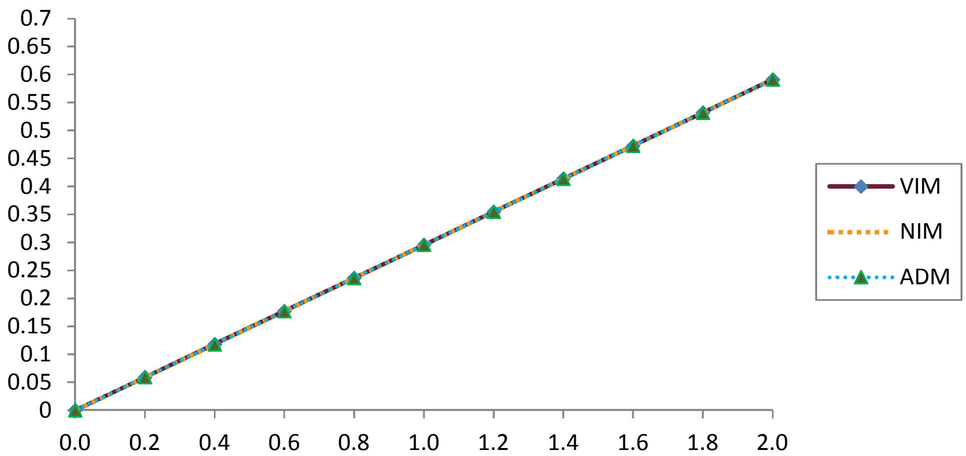

Figure 2: Comparison of the three methods for example 1 after 2nd iteration

Example 3

Consider the inhomogeneous linear Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 2\sin x \tag{29}$$

With the initial conditions

$$u(x,0) = \sin x, \quad u_t(x,0) = 1$$

and the exact solution is

$$u(x,t) = \sin x + \sin t$$

Following the procedures for the three methods after 3 iterations, the solution becomes

$$u(x,t) = \sin x + \sin t \tag{30}$$

Which is the exact solution

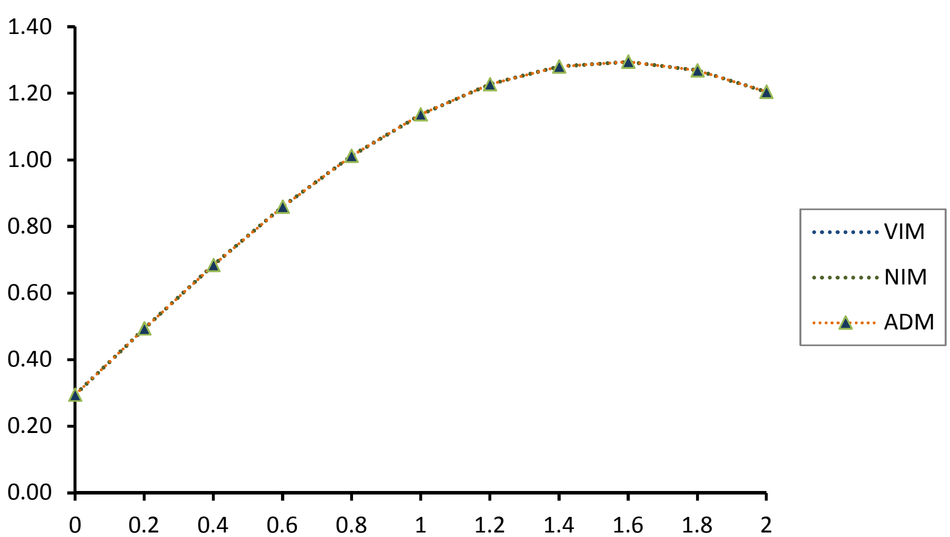

Figure 3: Comparison of the three methods for example 3 after 2nd iteration

Nonlinear Klein-Gordon Equations

Example 4

Consider the following Nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = x^2t^2 \tag{31}$$

With the boundary conditions

$u(x,0) = 0, \quad u_t(x,0) = x$

The exact solution is $u(x,t) = xt$

Following the procedures for the three methods after 3 iterations, the solution becomes

$$u(x,t) = xt \tag{32}$$

Which is the exact solution

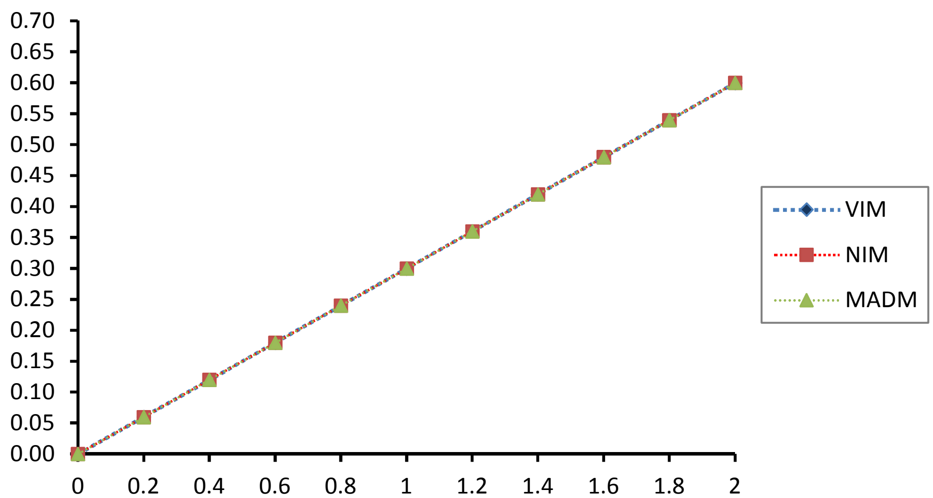

Figure 4: Comparison of the three methods for example 4 after 2nd iteration

Example 5

Given the following nonlinear inhomogeneous Klein-Gordon equation:

$$u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2 \tag{33}$$

With the initial conditions

$u(x,0) = 1, \quad u_t(x,0) = x$

The Exact solution is $u(x,t) = 1 + xt$

Following the procedures for the three methods after 3 iterations, the solution becomes

$$u(x,t) = 1 + xt \tag{34}$$

Which is the exact solution

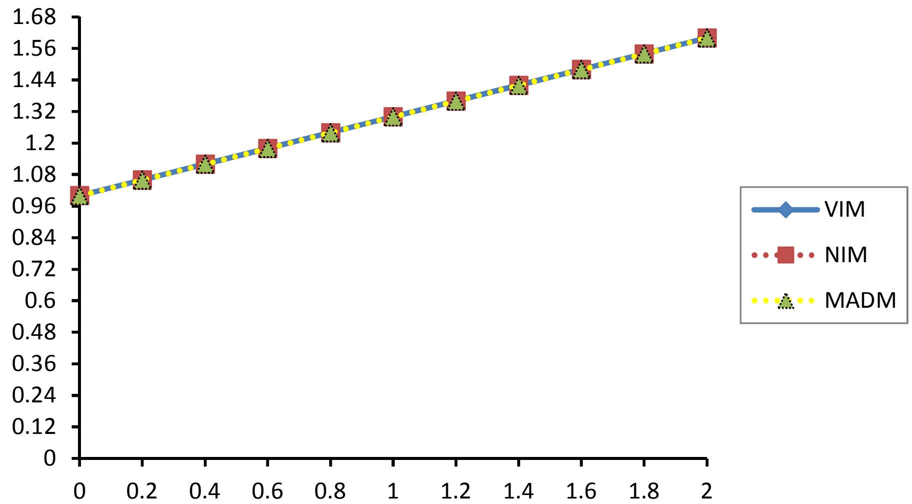

Figure 5: Comparison of the three methods for example 5 after 2nd iteration

Example 6

Given the following nonlinear inhomogeneous Klein-Gordon equation:

$$u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4t^4 \tag{35}$$

With the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0$$

And the exact solution is

$$u(x,t) = x^2t^2$$

Following the procedures for the three methods after 3 iterations, the solution becomes

$$u(x,t) = x^2t^2 \tag{36}$$

Which is the exact solution

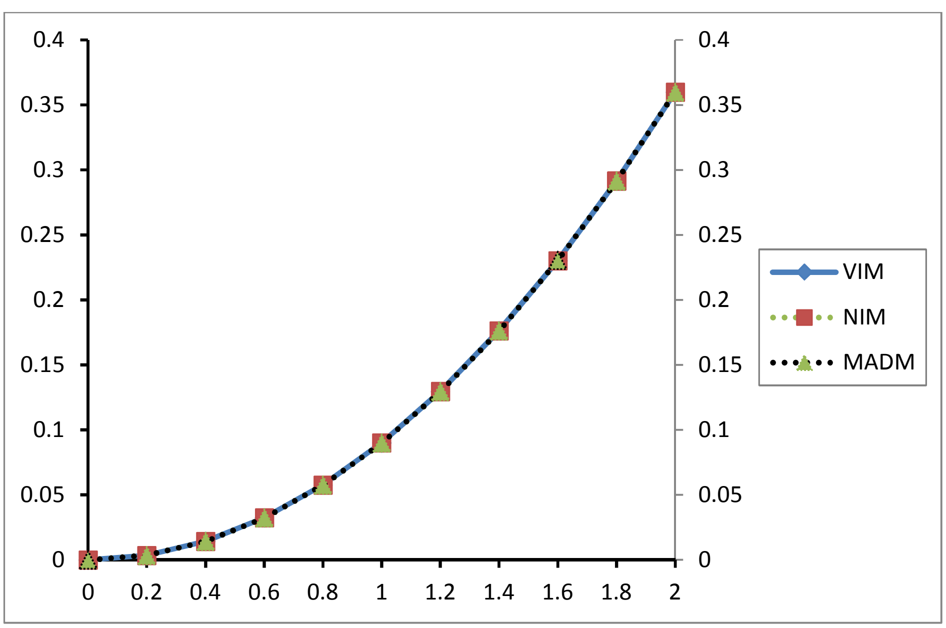

Figure 6: Comparison of the three methods for example 6 after 2nd iteration

CONCLUSION

In this paper, three methods, the Variational Iteration Method (VIM), the New Iterative Method (NIM) and the Adomian Decomposition Method (ADM) have been applied to Klein-Gordon equation and we compared the results of the three methods, the obtained results show that the three methods yielded the same results and they were excellent agreement with the exact solutions. It is capable to converge to exact solutions with fewest number of iterations.

Figures 1-6 justified that the methods are reliable and efficient and can be applied to linear and nonlinear equations of different parameters.

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