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TRANSVERSE RESPONSE OF AN ELASTICALLY RESTRAINED END DOUBLE-BEAM SYSTEM DUE TO A CONCENTRATED MOVING MASS

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ABSTRACT

In this paper, transverse response of a double beam system traversed by a concentrated moving mass is investigated. The system, which is elastically restrained at both ends, consists of two identical homogeneous parallel Euler-Bernoulli beams and are continuously interconnected with a viscoelastic Winkler-type layer. Of particular interest, is the effect of the moving load on the dynamic response of the system. The solution scheme deployed involves, using series variable separable method, modified Struble's method and differential transformation method (DTM). For the purpose of demonstrating the simplicity and efficiency of the technique, numerical examples are explored. The dynamic deflections of the beams are presented graphically and are found comparable with the existing results for the case of a moving force. The effect of various values of the mass of the moving load on the dynamic response of the beam are presented and discussed. The effects of the speed of the moving load, viscoelastic parameter and stiffness parameters are also examined

.

Keywords: Transverse response, Moving mass, Euler-Bernoulli beam, Viscoelastic Winkler-type layer, Dynamic deflection.

INTRODUCTION

The analysis of moving load problems is very crucial in structural dynamics. Over decades, there have been several investigations concerning the dynamic response of elastic beam structures under the influence of moving loads Fryba (1972), Gbadeyan and Oni (1995), Esmailzadeh and Ghorashi (1995), Seelig and Hoppmann (1964), Gbadeyan *et al* (2005), Stojanivic and Kozic (2012).. The effect of moving loads on engineering structure over which they travel is of great effect thereby causing an intense vibration due to high velocity. Several applications involving moving loads are found in the field of transportation. Various structures including road and highway bridges, railways, pipelines, rotating machines, tunnels and submarines have been subjected to either constant or distributed moving masses and in effect, the problem of assessing dynamic response of these structures due to moving masses has continue to generate a consistent research efforts towards mitigating against the potential hazard observed over time. The assumption of particular end conditions has considerably articulated various authors toward addressing most of the vibration problem of beam-type structures under the influence of consistent excitation phenomenon. In t he study presented by Oniszczuk (1999), free vibration of elastically connected double Euler-Bernoulli beam structure is considered. A set of partial differential equations describing the motion of the system is solved with the aid of classical Bernoulli-Fourier method. Oniszczuk (2008), developed a theory concerning free transverse elastically connected double-beam complex system. The classical Euler-Bernoulli-Fourier method is used to solve the final governing differential equations of motion. The analysis concerning the problem of a double beam system under an applied force is conducted by Oniszczuk, Z (2003). The two beams are connected together with a parallel distributed springs and dashpots. An exact method was presented to solve the vibration problem under harmonic excitation. The finite element analysis was conducted on a forced Timoshenko beams with different double end supports by Li and Hua (2016). The vibration modes and amplitude-frequency dependence for the forced vibration of the system was obtained. Abu-Hilal (2016), investigated the dynamic response of a simply supported double Euler-Bernoulli beam system under a moving constant load. The two beams are continuously connected with viscoelastic layer of springs and dashpots. The normalized deflections of the beams are presented in closed forms. In the study conducted by Miirzabeigy and Madoliat (2016), transverse free vibration of two parallel elastic beams which are partially connected together by elastic layer of Winkler type layer is investigated. The differential equations of motion for the system was simplified using differential transform method to yield natural frequencies and mode shapes. Effects of viscoelastic inner layer damping and Winkler-type elastic layer on the dynamic responses of a double-beam system is presented by Mohammedi and Nasirshoaibi (2008). Iterated modal expansion method was applied to obtain the forced vibration responses of the two beams sequel to the natural frequencies and mode shapes initially obtained from the free vibration analysis. Mirzabeigy *et al*, investigated free vibration of two parallel beams which are interconnected with variable stiffness of Winkler-type elastic layer. The fourth order partial differential equations of motion derived have been solved by applying differential transform method to obtain natural frequencies and normalized mode shapes of the system. In Li *et al* (2016) the effects of viscoelastic layer damping and Winkler-type layer on the dynamic responses of a beam system is investigated. A semi-analytical method developed is used to derive the natural frequencies and the corresponding mode shapes of vibration of the system with interconnected viscoelastic layer.

In this paper, the primary concern is the effect of the mass of the moving load on the dynamic responses of the two finite Euler-Bernoulli beams which are interconnected by a viscoelastic layer of springs and dashports. The beams are elastically restrained at both ends with translational and rotational springs. To the best knowledge of the authors of the present article, the configuration presented has not been reported in the previous studies. Having assumed a negligible noise effect, the influence of Gussian or non-gaussian noise and those of the output constraints has not been taken into account. In order to archive the desired objectives, a versatile solution scheme which is a modification of Gbadeyan, J.A. and Oni, S.T. (1995) and Andi, E.A. and Oni, S.T. (2014) is developed. This scheme involves a series variable separable method, Struble's asymptotic modification method and an iterative differential transform method.

THE STRUCTURAL AND MATHEMATICAL FORMULATION OF THE PROBLEM

The structural model of a double Euler-Bernoulli beam system under consideration is shown in Figure 1. This is composed of two undamped uniform finite beams which are interconnected by a concentrated viscoelastic Winkler-type layer of springs and dashpots.

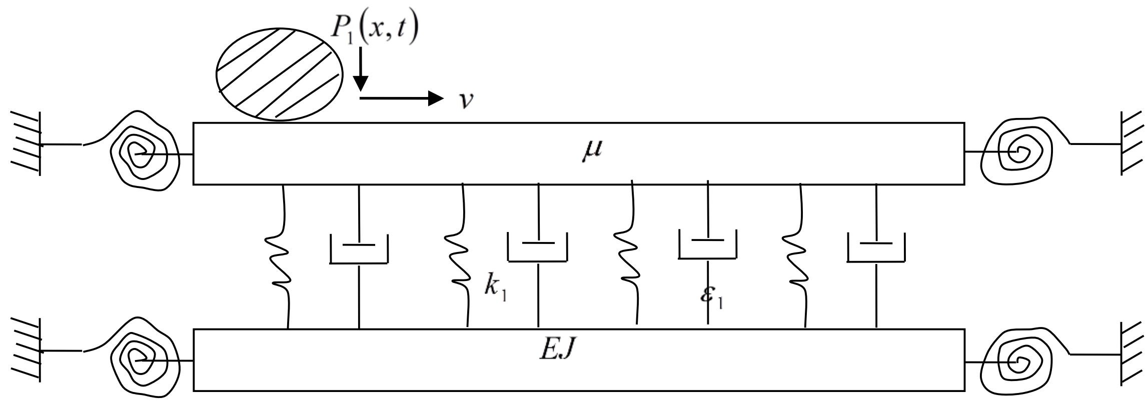

Figure 1: Double Euler-Bernoulli beam system with ends restrained : Mohammadi and Nasirshoabi (2015)

The beams are supported at both ends by translational and rotational springs at both ends. The coupled governing equations of motion of the vibrating beams are Mirzabegy *et al* (2016)

$$EI\frac{\partial^4 W_1(x,t)}{\partial x^4}+\mu\frac{\partial^2 W_1(x,t)}{\partial t^2}=P_1(x,t)+\left[k_1+\varepsilon_0\frac{\partial}{\partial t}\right][W_1(x,t)-W_2(x,t)] \tag{1}$$

$$EI\frac{\partial^4 W_2(x,t)}{\partial x^4}+\mu\frac{\partial^2 W_2(x,t)}{\partial t^2}=\left[k_1+\varepsilon_0\frac{\partial}{\partial t}\right][W_2(x,t)-W_1(x,t)] \tag{2}$$

where the dynamic concentrated moving load is defined as

$$P_1(x,t)=\left[M_L g-M_L\frac{\partial^2 W_1(x,t)}{\partial t^2}-2M_L v\frac{\partial^2 W_1(x,t)}{\partial t\partial x}-M_L v^2\frac{\partial^2 W_1(x,t)}{\partial x^2}\right]\partial(x-vt) \tag{3}$$

E and I are the Young's modulus and moment of inertia of the beams. $W_1(x,t)$ and $W_2(x,t)$ are the transverse response functions. x is the spatial coordinate along the beam length, t is the time, μ and L are the mass per unit length and length of the beams.v and M_L are the velocity and mass of the moving load. The stiffness of the constant viscoelastic Winkler inner layer is k_1. Acceleration due to gravitational force $g(9.81ms^{-2})$ while $\delta(x)$ is the Dirac delta function defined as,

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \tag{4}$$

An even function of Fourier cosine series is considered for $\delta(x - vt)$ and is expressed as [1, 2, 3]

$$\delta(x - vt) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L} cos\frac{n\pi x}{L} \tag{5}$$

The dynamic vibrating structure have elastically restrained end conditions written as [20, 25],

$$\frac{\partial^2 W_i(0,t)}{\partial x^2} = 0 = \frac{\partial^2 W_i(L,t)}{\partial x^2} \tag{6}$$

$$E_i I_i \frac{\partial^3 W_i'''(0,t)}{\partial x^3} = -k_1 W_i(0,t) \tag{7}$$

$$E_i I_i \frac{\partial^3 W_i(L,t)}{\partial x^3} = k_1 W_i(0,t) \tag{8}$$

where $i = 1, 2$, is representing the upper and lower beams respectively. The initial conditions in general form are

$$W_i(x,t)|_{t=0} = 0 = \frac{\partial W_i(x,t)}{\partial t}|_{t=0}, \quad i = 1, 2 \tag{9}$$

METHOD OF SOLUTION

In order to solve the vibrating dynamic equations (1) and (2), the use of certain scheme is employed. (i) Applying the series variable separable method which assume solutions of the form

$$W_1(x,t) = \sum_{m=1}^{\infty} \phi_m(m;t) V_m(x) \tag{10}$$

$$W_2(x,t) = \sum_{m=1}^{\infty} \beta_m(m;t) V_m(x) \tag{11}$$

on equations (1) and (2) respectively. Here, $\phi_m(m;t)$, $\beta_m(m;t)$ are the time functions and $V_m(x)$ is the mode shape function for the upper and lower beams defined as

$$V_m(x) = sin\frac{\lambda_m}{L}x + A_m\, cos\frac{\lambda_m}{L}x + B_m\, sinh\frac{\lambda_m}{L}x + C_m\, cosh\frac{\lambda_m}{L}x, \qquad m = 1, 2, ... \tag{12}$$

where A_m, B_m, C_m are constants and can be derived by applying any of the elastic classical end conditions. (ii) Introducing the modification of Struble's Asymptotic method to simplify the reduced coupled equations. (iii) Applying the differential transform method to solve the obtained transformed equations in step (ii).

Transformation Procedure

The fourth order partial differential equations (1) and (2) are transformed to a pair of coupled second order ordinary differential equations, first by introducing equations (10) and (11) into equations (1) and (2) to obtain,

$$EI\sum_{m=1}^{\infty} \phi_m(m;t) V_m^{iv}(x) + \mu \sum_{m=1}^{\infty} \phi_m(m;t) V_m(x) = P_1(x,t) + \left[k_1 + \varepsilon_0 \frac{\partial}{\partial t}\right]\left[\sum_{m=1}^{\infty} \phi_m(m;t) V_m - \sum_{m=1}^{\infty} \beta_m(m;t) V_m(x)\right] \tag{13}$$

$$EI\sum_{m=1}^{\infty}\beta_m\,(m;t)V_m^{iv}(x)+\mu\sum_{m=1}^{\infty}\ddot{\beta}_m\,(m;t)V_m(x)$$
$$=\left[k_1+\varepsilon_0\frac{\partial}{\partial t}\right]\left[\sum_{m=1}^{\infty}\beta_m\,(m;t)V_m(x)-\sum_{m=1}^{\infty}\varphi_m\,(m;t)V_m(x)\right] \quad (14)$$

The load function $P_1(x,t)$ which is further assumed can be expressed as

$$P_1(x,t)=\sum_{m=1}^{\infty}\psi_m\,(m;t)V_m(x) \quad (15)$$

where $\psi_m(m,t)$ is time function.
Introducing equation (10) into equation (3) for an arbitrary subscript k, the obtained equation is

$$P_1(x,t)=[M_Lg-M_L\sum_{k=1}^{\infty}\ddot{\phi}_k(k;t)V_k(x)-2M_Lv\sum_{k=1}^{\infty}\dot{\phi}_k(k;t)V_k^{'}(x)$$
$$-M_Lv^2\sum_{k=1}^{\infty}\phi_k(k;t)V_k^{''}(x)]\partial(x-vt) \quad (16)$$

In view of equation (5), a known normalized deflection function $V_j(x),\ j=1,2,3,\ldots$ is applied to the relation (15) and the obtained result is compared with relation (16). Furthermore, we note that the normalized deflection function $V_m(x),\ m=1,2,3,\ldots$ are orthonormal and the integration of the resulting equation along the length of the beam ($0\le x\le L$) is performed to yield the following time function:

$$\psi_m(m;t)=A_{11}+A_{12}+A_{13}+A_{14} \quad (17)$$

$$A_{11}=M_Lg\Delta_{10}(j)+\frac{2M_Lg}{L}\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}\Delta_{20}(j,k) \quad (18)$$

$$A_{12}=-\frac{M_L}{L}\sum_{k=1}^{\infty}\ddot{\phi}_k(k;t)\left[\Delta_1(j,k)+\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}\Delta_{1c}(j,k)\right] \quad (19)$$

$$A_{13}=-2\frac{M_Lv}{L}\sum_{k=1}^{\infty}\dot{\phi}_k(k;t)\left[\Delta_2(j,k)+2\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}\Delta_{2c}(j,k)\right] \quad (20)$$

$$A_{14}=-\frac{M_Lv^2}{L}\sum_{k=1}^{\infty}\phi_k(k;t)\left[\Delta_3(j,k)+2\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}\Delta_{3c}(j,k)\right] \quad (21)$$

$$\Delta_{10}(j)=\frac{1}{L}\int_0^L V_j(x)dx\,;\Delta_{20}(j,n)=\int_0^L cos\frac{n\pi x}{L}V_j(x)dx \quad (21a)$$

$$\Delta_1(j,k)=\int_0^L V_k(x)V_j(x)dx\,;\ \ \Delta_{1c}(j,k)=\int_0^L cos\frac{n\pi x}{L}V_k(x)V_j(x)dx \quad (21b)$$

$$\Delta_2(j,k)=\int_0^L V_k^{'}(x)V_j(x)dx\,;\ \ \Delta_{2c}(j,k)=\int_0^L cos\frac{n\pi x}{L}V_k^{'}(x)V_j(x)dx \quad (21c)$$

$$\Delta_3(j,k)=\int_0^L V_k^{''}(x)V_j(x)dx\,;\ \ \Delta_{3c}(j,k)=\int_0^L cos\frac{n\pi x}{L}V_k^{''}(x)V_j(x)dx \quad (21d)$$

The load function $P_1(x,t)$ is determined initially, by substituting equation (17) into equation (15) noting the relations (18) – (21) and (21a) – (21d). Hence,

$$P_1(x,t) = \sum_{m=1}^{\infty} V_m(x)[M_L g V_m(vt)$$

$$- \frac{M_L}{L}\sum_{k=1}^{\infty}\ddot{\phi}_k(k;t)\left[\Delta_1(j,k) + 2\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L}\Delta_{1c}(j,k)\right]$$
$$-2\frac{M_L v}{L}\sum_{k=1}^{\infty}\dot{\phi}_k(k;t)\left[\Delta_2(j,k) + 2\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L}\Delta_{2c}(j,k)\right]$$
$$-\frac{M_L v^2}{L}\sum_{k=1}^{\infty}\phi_k(k;t)\left[\Delta_3(j,k) + 2\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L}\Delta_{3c}(j,k)\right]\Bigg] \quad (22)$$

By substituting equation (22) into equation (13), we have

$$EI\sum_{m=1}^{\infty}\phi_m(m;t)V_m^{iv}(x) + \mu\sum_{m=1}^{\infty}\ddot{\phi}_m(m;t)V_m(x)$$
$$- k_1\sum_{m=1}^{\infty}\phi_m(m;t)V_m(x) + k_1\sum_{m=1}^{\infty}\beta_m(m;t)V_m(x) - \varepsilon_0\sum_{m=1}^{\infty}\dot{\phi}_m(m;t)V_m(x)$$
$$+ \varepsilon_0\sum_{m=1}^{\infty}\dot{\beta}_m(m;t)V_m(x) = \sum_{m=1}^{\infty}V_m(x)M_L g V_m(vt)$$
$$- \frac{M_L}{L}\sum_{k=1}^{\infty}\ddot{\phi}_k(k;t)\left[\Delta_1(j,k) + 2\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L}\Delta_{1c}(j,k)\right]$$
$$- 2\frac{M_L v}{L}\sum_{k=1}^{\infty}\dot{\phi}_k(k;t)\left[\Delta_2(j,k) + 2\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L}\Delta_{2c}(j,k)\right]$$
$$- \frac{M_L v^2}{L}\sum_{k-1}^{\infty}\phi_k(k;t)\left[\Delta_3(j,k) + 2\sum_{n-1}^{\infty} cos\frac{n\pi vt}{L}\Delta_{3c}(j,k)\right] \quad (23)$$

Based on previous results, it is remarked that for free vibration of a single Euler-Bernoulli beam structure, the following homogeneous equation holds.

$$EIV_m^{iv}(x) - \mu\omega_m^2 V_m(x) = 0 \quad (24)$$

$$\omega_m^2 = \frac{\lambda_m^4}{L^4}\frac{EI}{\mu} \quad (25)$$

such that ω_m is the mth natural frequency of the beam structure.
By applying equation (24) on equation (23) and after some algebraic simplifications, the obtained equation is

$$\omega_m^2\phi_m(m;t) + \ddot{\varphi}_m(m;t) - \frac{k_1}{\mu}\phi_m(m;t) + \frac{k_1}{\mu}\beta_m(m;t) - \frac{\varepsilon_0}{\mu}\dot{\phi}_m(m;t) + \frac{\varepsilon_0}{\mu}\dot{\beta}_m(m;t)$$
$$+\varepsilon_1\sum_{k=1}^{\infty}\dot{\phi}_k(k;t)\left[\Delta_1(j;k) + 2\sum_{n-1}^{\infty} cos\frac{n\pi vt}{L}\Delta_{1c}(j;k)\right]$$
$$+2\varepsilon_1 v\sum_{k=1}^{\infty}\dot{\phi}_k(k;t)\left[\Delta_2(j;k) + 2\sum_{n-1}^{\infty} cos\frac{n\pi vt}{L}\Delta_{2c}(j;k)\right]$$
$$+\varepsilon_1 v^2\sum_{k=1}^{\infty}\dot{\phi}_k(k;t)\left[\Delta_3(j;k) + 2\sum_{n-1}^{\infty} cos\frac{n\pi vt}{L}\Delta_{3c}(j;k)\right] = \varepsilon_1 g L V_m(vt) \quad (26)$$

$$\varepsilon_1 = \frac{M_L}{\mu L}\ (mass\ ratio) \tag{26a}$$

Applying the same argument thus far, on equation (14), the obtained result is,

$$\ddot{\beta}_m(m;t) + \omega_m^2 \beta_m(m;t) - \frac{k_1}{\mu}\beta_m(m;t) + \frac{k_1}{\mu}\phi_m(m;t) + \frac{\varepsilon_0}{\mu}\dot{\phi}_m(m;t) - \frac{\varepsilon_0}{\mu}\dot{\beta}_m(m;t) = 0 \tag{27}$$

Hence, equations (26) and (27) are the coupled transformed second order ordinary differential equations governing the lateral vibration of the double Euler-Bernoulli beam structure depicted in Figure 1, with a constant viscoelastic inner layer under a concentrated moving mass.

Simplification of the coupled second order ordinary differential equations

At the moment, it is remarked that obtaining the solution of equations (26) and (27) still appearing difficult, not only as a result of their high degree of coupleness, but also due to the presence of the coefficient of the terms representing the inertia of the moving load. These terms are functions of the independent variable t which is associated with equation (26), in particular. Hence, further simplification is conceived and thereby conducted on the reduced dynamic equations of motion. In order to accomplish this task, some procedural schemes are introduced. This includes decoupling equations (26) and (27) and obtaining a modified frequency which is due to the effect of the mass of the moving load.

Method of Decoupling Equations

The decoupling of equations (26) and (27) is attained initially by considering a dynamical system consisting of two disjointed Euler-Bernoulli beams such that, each is characterized with its elastic restrained ends. However, it is assumed that the first beam ($i = 1$) is acted upon by a concentrated moving mass while the second beam ($i = 2$) vibrates freely. To this end, equations (26) and (27) becomes

$$\ddot{\phi}_m(m;t) + \omega_m^2\phi_m(m;t) + \frac{M_L}{\mu L}\sum_{k=1}^{\infty}\ddot{\phi}_k(k;t)\left[\Delta_1(j,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}\Delta_{1c}(j,k)\right] + 2\varepsilon_1 v\sum_{k=1}^{\infty}\ddot{\phi}_k(k;t)\left[\Delta_2(j,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}\Delta_{2c}(j,k)\right] + \varepsilon_1 v^2\sum_{k=1}^{\infty}\ddot{\phi}_k(k;t)\left[\Delta_3(j,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}\Delta_{3c}(j,k)\right] = \varepsilon_1 gLV_m(vt) \tag{28}$$

$$\ddot{\beta}_m(m;t) + \omega_m^2\beta_m(m;t) = 0 \tag{29}$$

Method of Obtaining Modified Frequency

Due to the inherent difficulty in the solution of equations (28) and (29), an approximate analytical scheme which is a modified asymptotic method of Stuble's technique [1,3] is applied to equation (28), in particular. This involves obtaining a modified frequency due to the inertia effect of the mass of the moving load so that each differential operator in equation (28) is replaced by an equivalent operator defined by the modified frequency. Hence, the mass ratio of the moving load to its length is demoted by ε_1, and a small parameter λ, is introduced such that

$$\lambda = \frac{\varepsilon_1}{1+\varepsilon_1} < 1 \tag{30}$$

Obviously, one can easily deduce that

$$\varepsilon_1 = \lambda + 0(\lambda^2) \tag{31}$$

According to Struble's technique [1,3], the task of obtaining the desired modified frequency involves considering the following first approximate solution to the homogenous part of equation (28).

$$\phi_m(m;t) = \theta(m;t)\cos[\omega_m t - \alpha_m(m;t)] + \sum_{r=1}^{\bar{N}} \lambda^r \phi_r(m;t) + O\left(\lambda^{\bar{N}+1}\right) \tag{32}$$

where $0 < \bar{N} < \infty$, $\theta(m;t)$ and $\alpha(m;t)$ are slowly time varying functions.
Now, apply equation (32), its first and second derivatives to the said homogeneous part and taking into account equation (31), the obtained equation after some algebraic simplification is

$$\begin{aligned} &2\omega_m\theta(m;t)\dot{\alpha}(m;t)\cos[\omega_m t - \alpha_m(m;t)] - 2\omega_m\theta(m;t)\sin[\omega_m t - \alpha_m(m;t)] \\ &- 2\lambda v\omega_m\theta(m;t)\Delta_2(j,m)\sin[\omega_m t - \alpha_m(m;t)] \\ &- 4\lambda v\omega_m\theta(m;t)\Delta_2(j,m)\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}\sin[\omega_m t - \alpha_m(m;t)] \\ &- \lambda\omega_m^2\theta(m;t)\Delta_1(j,m)\cos[\omega_m t - \alpha_m(m;t)] \\ &- 2\lambda\omega_m^2\theta(m;t)\Delta_{1c}(j,m)\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}\cos[\omega_m t - \alpha_m(m;t)] \\ &+ \lambda v^2\Delta_3(j,m)\theta(m;t)\cos[\omega_m t - \alpha_m(m;t)] \\ &+ 2\lambda v^2\Delta_3(j,m)\theta(m;t)\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}\cos[\omega_m t - \alpha_m(m;t)] = 0 \end{aligned} \tag{33}$$

In obtaining equation (33), all the terms in λ^2 and higher powers of λ have been considered as negligibly unimportant. Thus, the corresponding variational equations derived from equation (33) are

$$-2\omega_m\dot{\theta}(m;t) - 2\lambda v\omega_m\theta(m;t)\Delta_2(j,m) = 0 \tag{34}$$

$$2\omega_m\dot{\alpha}_m(m;t) - \lambda\omega_m^2\Delta_1(j,m) + \lambda v^2\Delta_3(j,m) = 0 \tag{35}$$

Solving equations (34) and (35) yields,

$$\theta(m;t) = C_0 e^{q_0 t} \tag{36}$$
$$\alpha_m(m;t) = C_0 e^{q_0 t}\cos(\beta_m t - \tau_m) \tag{37}$$

where $q_0 = \lambda v\Delta_2(j,m), C_0, \tau_m$ are constants and

$$\beta_m = \omega_m\left[1 - \frac{\lambda}{2}\left(\Delta_1(j,m) - \frac{v^2\Delta_3(j,m)}{\omega_m^2}\right)\right] \tag{38}$$

is the modified frequency which corresponds to the frequency of the free system involving the effect of the mass of the moving load. In line with the technique of Struble [1,3], equation (28) reduces to

$$\begin{aligned} &\ddot{\phi}_m(m;t) + \beta_m^2\phi_m(m;t) - \frac{k_1}{\mu}\phi_m(m;t) + \frac{k_1}{\mu}\beta_m(m;t) - \frac{\varepsilon_0}{\mu}\dot{\varphi}_m(m;t) \\ &\qquad + \frac{\varepsilon_0}{\mu}\dot{\beta}_m(m;t) = \varepsilon_1 gLV_m(vt) \end{aligned} \tag{39}$$

Using the same argument, equation (14) which is for the second beam $(i = 2)$ has been simplified and reduces to

$$\begin{aligned} &\ddot{\beta}_m(m;t) + \omega_m^2\beta_m(m;t) - \frac{k_1}{\mu}\beta_m(m;t) + \frac{k_1}{\mu}\phi_m(m;t) \\ &\qquad + \frac{\varepsilon_0}{\mu}\dot{\phi}_m(m;t) - \frac{\varepsilon_0}{\mu}\dot{\beta}_m(m;t) = 0 \end{aligned} \tag{40}$$

Hence, equations (39) and (40) are the reduced transformed second order ordinary differential equations of the original system whose inner layer have been retained.

DIFFERENTIAL TRANSFORM METHOD

In order to solve the dynamic equations (39) and (40), a known iterative scheme called differential transform method (DTM) attributed to Zhou (1986) which has been applied by several authors Raslan, et al. (2012), Abel-Halim and Erturk (2009) is hereby instroduced. This scheme involves considering a continuous k[th] derivative of a time function $\phi_m(m;t)$ such that,

$$\bar{\phi}_m(k) = \frac{1}{k!}\left[\frac{d^k\phi_m(m,t)}{dt^k}\right]_{t=t_0} \tag{41}$$

The inverse differential transform of equation (41) is defined as

$$\phi_m(m,t) = \sum_{k=0}^{\infty}\bar{\phi}_m(k)(t-t_0)^k \tag{42}$$

Comparing equations (41) and (42) yields

$$\bar{\phi}_m(m,t) = \sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\left[\frac{d^k\bar{\phi}_m(m,t)}{dt^k}\right]_{t=t_0} \tag{43}$$

It has been established that in practical applications, the series in equation (43) is finite. Hence, it can be expressed as

$$\phi_m(m,t) = \sum_{k=0}^{m}\bar{\phi}_m(k)t^k \tag{44}$$

such that $\sum_{k=m+1}^{\infty}\bar{\phi}_m(k)t^k$ is considered to be negligibly small.

Tables 1 and 2 below consist of a summary of some of the relationships [28, 29] between the original function and the transformed function when $t_0 = 0$.

Table 1: Theorems of DTM for Equations of Motion

Original function	**T-function**
$w(t) = u(t) \pm v(t)$	$\overline{W}(k) = \overline{U}(k) \pm \overline{V}(k)$
$w(t) = cu(t)$	$\overline{W}(k) = c\overline{U}(k)$
$w(t) = \dfrac{du(t)}{dt}$	$\overline{W}(k) = (k+1)\overline{U}(k+1)$
$w(t) = \dfrac{d^n u(t)}{dt^n}$	$\overline{W}(k) = (k+1)(k+2)...(k+n-1)(k+n)\overline{U}(k+n)$
$w(t) = u(t)v(t)$	$\overline{W}(k) = \sum_{n=0}^{k}\overline{U}(n)\overline{V}(k-n)$
$w(t) = \sin at$	$\overline{W}(k) = \dfrac{1}{k!}a^k \sin\left(\dfrac{k\pi}{2}\right)$
$w(t) = \cos at$	$\overline{W}(k) = \dfrac{1}{k!}a^k \cos\left(\dfrac{k\pi}{2}\right)$
$w(t) = \sinh at$	$\overline{W}(k) = \dfrac{1}{2k!}[(a)^k - (-a)^k]$
$w(t) = \cosh at$	$\overline{W}(k) = \dfrac{1}{2k}[(a)^k + (-a)^k]$

Table 2: Theorem of DTM for Boundary Conditions

Original $BC(t=0)$	$T-BC(x=0)$	**Original** $BC(t=1)$	$T-BC(t=1)$
$w(0)=0$	$\overline{W}(0)=0$	$w(1)=0$	$\sum_{k=0}^{\infty}\overline{W}(k)=0$
$\frac{dw(0)}{dx}=0$	$\overline{W}(1)=0$	$\frac{dw(1)}{dx}=0$	$\sum_{k=0}^{\infty}k\overline{W}(k)=0$
$\frac{d^2w(0)}{dx^2}=0$	$\overline{W}(2)=0$	$\frac{d^2w(1)}{dx^2}=0$	$\sum_{k=0}^{\infty}k(k-1)\overline{W}(k)=0$
$\frac{d^2w(0)}{dx^3}=0$	$\overline{W}(3)=0$	$\frac{d^2w(1)}{dx^3}=0$	$\sum_{k=0}^{\infty}k(k-1)(k-2)\overline{W}(k)=0$

In view of equation (12), the results in Tables 1 and 2 are applied on the transformed equations (39) and (40) to yield the flowing recurrence relations.

$$\bar{\phi}_m(k+2)=\frac{1}{(k+1)(k+2)}\Bigg[D_1\left(\frac{\eta_m^k}{k!}sin\left(\frac{k\pi}{2}\right)+\frac{A_m}{k!}(\eta_m)cos\left(\frac{k\pi}{2}\right)\right.$$
$$\left.+\frac{B_m}{2k!}[(\eta_m)^4-(\eta_m)^k]+\frac{C_m}{2k!}[(\eta_m)^k+(-\eta_m)^k]\right)-\beta^2\bar{\phi}_m(k)$$
$$+\frac{k_1}{\mu}\bar{\phi}_m(k)-\frac{\varepsilon_0}{\mu}(k+1)\bar{\beta}_m(k+1)-\frac{k_1}{\mu}\bar{\beta}_m(k)+\frac{\varepsilon_0}{\mu}(k+1)\bar{\phi}_m(k+1)\Bigg] \quad (45)$$

$$\bar{\beta}_m(k+2)=\frac{1}{(k+1)(k+2)}\Bigg[-\omega_m^2\bar{\beta}_m(k)+\frac{k_1}{\mu}\bar{\beta}_m(k)$$
$$-\frac{\varepsilon_0}{\mu}(k+1)\bar{\beta}_m(k+1)+\frac{k_1}{\mu}\bar{\phi}_m(k)+\frac{\varepsilon_0}{\mu}(k+1)\bar{\phi}_m(k+1)\Bigg] \quad (46)$$

$$\eta_m=\frac{\lambda_m v}{L}$$

The transformed initial conditions at $t_0=0$ are,

$$\bar{\phi}_m(0)=0=\bar{\phi}_m(1) \quad (47)$$
$$\bar{\beta}_m(0)=0=\bar{\beta}_m(1) \quad (48)$$

Applying $k=1,2,3,\ldots$ on equations (45) and (46) using "MAPLE 18", the corresponding results are

$$\bar{\phi}_m(2)=\frac{D_1}{2}(A_m+C_m)+\frac{\varepsilon_0}{\mu}(A_m+C_m) \quad (49)$$
$$\bar{\beta}_m(2)=0 \quad (50)$$
$$\bar{\phi}_m(3)=\frac{D_1\eta_m}{6}(1+B_m) \quad (51)$$

$$\bar{\beta}_m(3)=\frac{D_1\varepsilon_0}{6\mu}(A_m+C_m) \quad (52)$$

$$\bar{\phi}_m(4)=\frac{D_1}{4!}\left[(\eta_m^2-\beta_m^2)C_m-(\eta_m^2-\beta_m^2)A_m+2\frac{\varepsilon_0^2}{\mu^2}(A_m+C_m)+\frac{\varepsilon_0}{\mu}\eta_m(1+B_m)\right] \quad (53)$$

$$\bar{\beta}_m(4) = \frac{D_1}{4!}\left[-2\frac{\varepsilon_0^2}{\mu^2}(A_m + C_m) - \frac{\varepsilon_0}{\mu}\eta_m(1 + B_m) \quad - \frac{k_1}{\mu}(A_m + C_m)\right] \tag{54}$$

$$\bar{\phi}_m(5) = \frac{D_1}{5!\,4!}\left[\beta_m^2\frac{\varepsilon_0}{\mu}(A_m + C_m) - \frac{k_1\varepsilon_0}{\mu^2}(A_m + C_m) - \frac{\varepsilon_0}{\mu}(A_m + C_m) \right.$$
$$\left. - \frac{\varepsilon_0}{\mu}(\eta_m^2 - \beta_m^2)C_m - \frac{\varepsilon_0}{\mu}(\eta_m^2 - \beta_m^2)A_m - \frac{\varepsilon_0}{\mu^2}\eta_m^2(1 - B_m)\right] \tag{55}$$

$$\bar{\beta}_m(5) = \frac{D_1}{5!}\left[\beta_m^2\frac{\varepsilon_0}{\mu}(A_m + C_m) - \frac{k_1\varepsilon_0}{\mu^2}(A_m + C_m) - \frac{\varepsilon_0}{\mu}(\eta_m^2 - \beta_m^2)C_m \right.$$
$$\left. - \frac{\varepsilon_0}{\mu}(\eta_m^2 + \beta_m^2) - \frac{\varepsilon_0 k_1}{\mu^2}\eta_m(1 + B_m)\right] \tag{56}$$

$$\bar{\phi}_m(6) = \frac{D_1}{6!}\left[\eta_m^4(A_m + C_m) + \left(\frac{k_1}{\mu} - \beta_m^2\right)(\eta_m^2 + \beta_m^2)C_m + \left(\beta_m^2 - \frac{k_1}{\mu}\right)\right.$$
$$- (\eta_m^2 + \beta_m^2)A_m + \frac{2k_1\varepsilon_0^2}{\mu^3}(A_m + C_m) + \left(2\frac{\varepsilon_0^2}{\mu^2} - \frac{\varepsilon_0\beta_m^2}{\mu}\right)(1 - B_m)\eta_m$$
$$\left. - \left(\frac{k_1}{\mu} - \frac{\varepsilon_0^2}{\mu^2}\right)(A_m + C_m) + \left(\frac{2k_1^2}{\mu^2} - \frac{\varepsilon_0^4}{\mu^4}\right)(A_m + C_m) \right] \tag{57}$$

$$\bar{\beta}_m(6) = \frac{D_1}{6!\,(\eta_m^2 + \beta_m^2)}\left[\omega_m^2\left(\frac{\varepsilon_0^2}{\mu^2} - \frac{k_1}{\mu}\right)(\eta_m^2 + \beta_m^2)(A_m + C_m)\right.$$
$$- \frac{\omega_m^2\varepsilon_0\eta_m^2}{\mu}(\eta_m^2 + \beta_m^2)(1 + B_m) + \frac{k_1\varepsilon_0^2}{\mu^3}(\eta_m^2 + \beta_m^2)(A_m + C_m)$$
$$+ \left(\frac{\varepsilon_0^2}{\mu^2} + \frac{k_1}{\mu}\right)(\eta_m^2 + \beta_m^2)^2 C_m + \left(\frac{\varepsilon_0^2}{\mu^2} + \frac{k_1}{\mu}\right)(\eta_m^2 + \beta_m^2)(\eta_m^2 - \beta_m^2)A_m$$
$$\left. - \left(\frac{\varepsilon_0^4}{\mu^4} + \frac{k_1\varepsilon_0}{\mu^2} + \frac{2\varepsilon_0}{\mu}\right)(\eta_m^2 + \beta_m^2)(A_m + C_m)\right] \tag{58}$$

where $\mu = \eta_m^2$

By applying the inverse differential transform on equation (44) and noting equation (47) - (58), the obtained dynamic lateral deflection equations for the upper and lower Euler-Bernoulli beam system under a concentrated moving mass is

$$W_1(x;t) = \sum_{m=1}^{\infty}\frac{D_1}{(\eta_m^2 - \beta_m^2)}\left\{\frac{(\eta_m^2 - \beta_m^2)(A_m + C_m)}{2!}t^2 + \frac{\eta_m^2(\eta_m^2 - \beta_m^2)(1 + B_m)}{3!}t^3\right.$$
$$+ \frac{(\eta_m^2 - \beta_m^2)^2}{4!}\left[\frac{C_m}{(\eta_m^2 - \beta_m^2)} - \frac{(\eta_m^2 + \beta_m^2)A_m}{(\eta_m^2 - \beta_m^2)} + \frac{k_1}{\mu}\frac{(A_m + C_m)}{(\eta_m^2 - \beta_m^2)}\right.$$
$$\left. + 2\frac{\varepsilon_0^2}{\mu^2}\frac{(A_m + C_m)}{(\eta_m^2 - \beta_m^2)} + \frac{\varepsilon_0\eta_m}{\mu}\frac{(1 + B_m)}{(\eta_m^2 - \beta_m^2)}\right]t^4 + \frac{(\eta_m^2 - \beta_m^2)}{5!}\left[\frac{\beta_m^2\varepsilon_0}{\mu}\frac{(A_m + C_m)}{(\eta_m^2 - \beta_m^2)}\right.$$
$$- \frac{k_1\varepsilon_0}{\mu^2}\frac{(A_m + C_m)}{(\eta_m^2 - \beta_m^2)} - \frac{\varepsilon_0^3\eta_m}{\mu^3}\frac{(A_m + C_m)}{(\eta_m^2 - \beta_m^2)} - \frac{\varepsilon_0}{\mu}\frac{C_m(\eta_m^2 - \beta_m^2)}{(\eta_m^2 - \beta_m^2)} - \frac{\varepsilon_0}{\mu}\frac{A_m(\eta_m^2 + \beta_m^2)}{(\eta_m^2 - \beta_m^2)}$$
$$\left. - \frac{\varepsilon_0 k_1\eta_m}{\mu^2}\frac{(1 + B_m)}{(\eta_m^2 - \beta_m^2)}\right]t^5 + \frac{(\eta_m^2 - \beta_m^2)^2}{6!}\left[\frac{\eta_m^4(A_m + C_m)}{(\eta_m^2 - \beta_m^2)}\right.$$

$$+\left(\frac{k_1}{\mu}-\beta_m^2\right)\frac{(\eta_m^2-\beta_m^2)}{(\eta_m^2-\beta_m^2)}C_m+\left(\beta_m^2+\frac{k_1}{\mu}\right)(\eta_m^2+\beta_m^2)\frac{A_m}{(\eta_m^2-\beta_m^2)}$$

$$+\frac{2k_1\varepsilon_0^2}{\mu^3}\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}+\left(2\frac{\varepsilon_0^2}{\mu^2}-\frac{\varepsilon_0\beta_m^2}{\mu}\right)\frac{\eta_m(1+\beta_m)}{(\eta_m^2-\beta_m^2)}$$

$$-\left(\frac{k_1}{\mu}-\frac{\varepsilon_0^2}{\mu^2}\right)\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}+\left(\frac{2k_1^2}{\mu^2}-\frac{\varepsilon_0^4}{\mu^4}\right)\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}\Bigg]t^6+\cdots\Bigg\}$$

$$\times\left[\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L}\right] \tag{59}$$

$$W_2(x;t)=\sum_{m=1}^{\infty}\frac{C_1}{(\eta_m^2-\beta_m^2)}\Bigg\{\frac{\varepsilon_0^2}{3!\,\mu^2}(\eta_m^2-\beta_m^2)(A_m+C_m)t^3$$

$$+\frac{(\eta_m^2-\beta_m^2)}{4!}\left[-\frac{\varepsilon_0^2}{\mu^2}\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}-\frac{\varepsilon_0}{\mu}\frac{(1+B_m)}{(\eta_m^2-\beta_m^2)}-\frac{k_1}{\mu}\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}\right]t^4-\frac{(\eta_m^2-\beta_m^2)^2}{5!}\Bigg[\frac{\beta_m^2\varepsilon_0}{\mu}\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}$$

$$-\frac{k_1\varepsilon_0}{\mu^2}\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}-\frac{\varepsilon_0}{\mu}\frac{(\eta_m^2-\beta_m^2)}{(\eta_m^2-\beta_m^2)}C_m-\frac{\varepsilon_0}{\mu}\frac{(\eta_m^2-\beta_m^2)}{(\eta_m^2-\beta_m^2)}A_m$$

$$-\frac{\varepsilon_0\,k_1}{\mu^2}\frac{\eta_m(1+B_m)}{(\eta_m^2-\beta_m^2)}\Bigg]t^5+\frac{(\eta_m^2-\beta_m^2)}{5!}\Bigg[\omega_m^2\left(\frac{\varepsilon_0^2}{\mu^2}-\frac{k_1}{\mu}\right)\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}$$

$$-\frac{\omega_m^2\varepsilon_0^2\,\eta_m^2}{\mu^2}\frac{(1+B_m)}{(\eta_m^2-\beta_m^2)}+\frac{k_1\varepsilon_0^2}{\mu^3}\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}+\left(\frac{\varepsilon_0^2}{\mu^2}+\frac{k_1}{\mu}\right)(\eta_m^2-\beta_m^2)^2\frac{C_m}{(\eta_m^2-\beta_m^2)}$$

$$+\left(\frac{\varepsilon_0^2}{\mu^2}+\frac{k_1}{\mu}\right)\frac{(\eta_m^2+\beta_m^2)}{(\eta_m^2-\beta_m^2)}A_m-\left(\frac{\varepsilon_0^4}{\mu^4}-\frac{k_1\varepsilon_0}{\mu^2}+\frac{2\varepsilon_0}{\mu}\right)\frac{(A_m+C_m)}{(\eta_m^2-\beta_m^2)}\Bigg]t^6+\cdots\Bigg\}$$

$$\times\left[\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L}\right] \tag{60}$$

ILLUSTRATIVE EXAMPLE OF AN ELASTICALLY RESTRAINED DOUBLE EULER-BERNOULLI BEAM SYSTEM

The theory discussed thus far, is for general boundary conditions. In this section, the system considered consists of two finite Euler-Bernoulli beams which are elastically restrained at both ends and are interconnected by a viscoelastic layer of springs and dashports. For such a system, the boundary conditions due to equations (6) – (8) are:

$$W_1(0,t)=0=W_1''(L,t) \tag{61}$$
$$W_2(0,t)=0=W_2''(L,t) \tag{62}$$
$$W_1'''(0,t)=C_{01}W_1(0,t) \tag{63}$$
$$W_2'''(0,t)=C_{02}W_2(0,t) \tag{64}$$
$$W_1'''(L,t)=C_{01}W_1(L,t) \tag{65}$$
$$W_2'''(L,t)=C_{02}W_2(L,t) \tag{66}$$

where $C_{01}=\frac{-k_0}{E_1I_1}$ and $C_{02}=\frac{k_0}{E_2I_2}$

The corresponding mode shape function, $V_m(x)$, have

$$V_m''(0) = 0 = V_m''(L) \tag{67}$$
$$V_m'''(0) = C_{01}V_m(L) \tag{68}$$
$$V_m'''(0) = C_{02}V_m(L) \tag{69}$$

as well as

$$V_r''(0) = 0 = V_r''(L) \tag{70}$$
$$V_r'''(0) = C_{01}V_r(L) \tag{71}$$
$$V_r'''(0) = C_{02}V_r'''(L) \tag{72}$$

By applying equations (67) – (69) on equation (12), we have

$$A_m = C_m = -\frac{sin\lambda_m}{cos\lambda_m}, \quad B_m = \frac{sin\lambda_m}{cos\lambda_m} \cdot \frac{cosh\lambda_m}{sinh\lambda_m} \tag{73}$$

Hence, the particular mode shape function, $V_m(x)$, for the system under consideration is

$$V_m(x) = \left(\sigma_m^* sinh\frac{\lambda_m}{L}x + sin\frac{\lambda_m}{L}x\right) + \sigma_m\left(cosh\frac{\lambda_m}{L}x - cos\frac{\lambda_m}{L}x\right) \tag{74}$$

where

$$\sigma_m^* = \frac{sinh\,\lambda_m}{cos\,\lambda_m} \cdot \frac{cosh\lambda_m}{sinh\lambda_m}, \qquad \sigma_m = \frac{sin\lambda_m}{cos\lambda_m} \tag{75}$$

The roots of the corresponding frequency equation [2] are:

$$\lambda_1 = 4.73004, \qquad \lambda_2 = 7.85320, \quad \lambda_3 = 10.99561, \dots \tag{76}$$

while the corresponding initial conditions due to equations (9) remain valid.
Now, two limiting cases of the forced vibrating problem which are associated with equations (39) and (40) are considered. This include,

I. Moving mass problem of a double finite Euler-Bernoulli beam system which is elastically restrained at both ends and interconnected by a visco elastic layer with the moving load inertia effect taking into account.

II. Moving force problem of a double finite Euler-Bernoulli beam system which is elastically restrained at both ends and interconnected by a viscoelastic layer and traversed by a moving load whose inertia effect is negligible.

In view of equations (39) and (40), the pair of the transformed governing equations of motion for the case I above is

$$\ddot{\phi}_{mm}(m;t) + \beta_{mm}^2\,\phi_{mm}(m;t) - \frac{k_1}{\mu}\phi_{mm}(m;t) + \frac{k_1}{\mu}\beta_{mm}(m;t) - \frac{\varepsilon_0}{\mu}\dot{\phi}_{mm}(m;t) + \frac{\varepsilon_0}{\mu}\dot{\beta}_{mm}(m;t) = D_1 V_{mm}(vt) \tag{77}$$

$$\ddot{\beta}_{mm}(m;t) + \beta_{mm}^2\,\phi_{mm}(m;t) - \frac{k_1}{\mu}\beta_{mm}(m;t) + \frac{k_1}{\mu}\phi_{mm}(m;t) + \frac{\varepsilon_0}{\mu}\dot{\phi}_{mm}(m;t) - \frac{\varepsilon_0}{\mu}\dot{\beta}_{mm}(m;t) = 0 \tag{78}$$

where,

$$D_1 = \varepsilon_1 gL\,;\; \beta_{mm}^2 = \frac{\omega_m}{2}\left[2 - \frac{2\lambda\left(\omega_m^2 - \frac{v^2m^2\pi^2}{L^2}\right)}{\omega_{mf}^2}\right] \tag{79}$$

The corresponding transformed governing equations for the case II are

$$\ddot{\phi}_{mf}(m;t) + \omega_{mf}^2\,\phi_{mf}(m;t) - \frac{k_1}{\mu}\beta_{mf}(m;t) + \frac{k_1}{\mu}\beta_{mf}(m;t)$$

$$-\frac{\varepsilon_0}{\mu}\dot{\phi}_{mf}(m;t)+\frac{\varepsilon_0}{\mu}\dot{\beta}_{mf}(m;t)=C_1V_m(vt) \tag{80}$$

$$\ddot{\beta}_{mf}(m;t)+\omega_{mf}^2\beta_{mf}(m;t)-\frac{k_1}{\mu}\beta_{mf}(m;t)+\frac{k_1}{\mu}\phi_{mf}(m;t)$$
$$+\frac{\varepsilon_0}{\mu}\dot{\phi}_{mf}(m;t)-\frac{\varepsilon_0}{\mu}\dot{\beta}_{mf}(m;t)=0 \tag{81}$$

where,

$$C_1=\frac{M_Lg}{\mu} \tag{80a}$$

Solving the resulting equations (77) and (78) by introducing the particular equation (74), and applying DTM in conjunction with the initial conditions, the obtained equations are

$$W_{1mm}(x,t)=\sum_{m=1}^{\infty}\frac{D_1}{(\eta_m^2-\beta_{mm}^2)}\left\{\frac{\frac{3}{2}\sigma_m(\eta_m^2-\beta_{mm}^2)}{2!}t^2-\frac{(\eta_m^2-\beta_{mm}^2)}{3!}\left[\sigma_m^*\eta_m+\eta_m\frac{3}{2}\frac{\varepsilon_0}{2\mu}\sigma_m\right]t^3\right.$$

$$+\frac{(\eta_m^2-\beta_m^2)^2}{4!}\left[\frac{\frac{\sigma_m}{2}(\eta_m^2+\beta_m^2)}{(\eta_m^2-\beta_{mm}^2)}-\frac{2\sigma_m\beta_{mm}^2}{(\eta_m^2-\beta_{mm}^2)}+\frac{\frac{3}{2}\frac{k_1}{\mu}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}+\frac{3\frac{\varepsilon_0^2}{\mu^2}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}\right.$$

$$\left.+\frac{\frac{\varepsilon_0}{\mu}\sigma_m^*\eta_m}{(\eta_m^2-\beta_{mm}^2)}+\frac{\frac{\varepsilon_0}{\mu}\eta_m}{(\eta_m^2-\beta_{mm}^2)}\right]t^4+\frac{(\eta_m^2-\beta_m^2)^2}{5!}\left[\sigma_m^*\eta_m-\frac{\eta_m(\eta_m^2+\beta_m^2)}{(\eta_m^2-\beta_{mm}^2)}\right.$$

$$-\frac{\beta_m^2\sigma_m^*\eta_m}{(\eta_m^2-\beta_{mm}^2)}-\frac{\frac{3}{2}\beta_{mm}^2\frac{\varepsilon_0}{\mu}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}+\frac{\frac{k_1}{\mu}\eta_m(1+\sigma_m^*)}{(\eta_m^2-\beta_{mm}^2)}+\frac{6\frac{k_1\varepsilon_0}{\mu^2}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}$$

$$+\frac{6\frac{\varepsilon_0^3}{\mu^3}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}-\frac{\frac{\varepsilon_0^2}{\mu^2}\eta_m(1+\sigma_m^*)}{(\eta_m^2-\beta_{mm}^2)}+\frac{\frac{1}{2}\sigma_m\frac{\varepsilon_0}{\mu}(\eta_m^2-\beta_{mm}^2)}{(\eta_m^2-\beta_{mm}^2)}$$

$$\left.\left.-\frac{2\frac{\varepsilon_0}{\mu}\sigma_m\beta_{mm}^2}{(\eta_m^2-\beta_{mm}^2)}-\frac{\frac{\varepsilon_0^2}{\mu^2}\sigma_m^*\eta_m}{(\eta_m^2-\beta_{mm}^2)}-\frac{\frac{\varepsilon_0^2}{\mu^2}\eta_m}{(\eta_m^2-\beta_{mm}^2)}\right]t^5+\cdots\right\}$$
$$\left(\sigma_m^*sinh\frac{\lambda_m}{L}x+sinh\frac{\lambda_m}{L}+\sigma_m cosh\frac{\lambda_m}{L}-\sigma_m cos\frac{\lambda_m}{L}x\right) \tag{82}$$

$$W_{2mm}(x,t)=\sum_{m=1}^{\infty}\frac{D_1}{(\eta_m^2-\beta_{mm}^2)}\left\{-\frac{\frac{3}{2}\frac{\varepsilon_0}{\mu}\sigma_m(\eta_m^2-\beta_m^2)}{3!}t^3+\frac{\sigma_m(\eta_m^2-\beta_m^2)^2}{4!}\left[\frac{\frac{\varepsilon_0}{\mu}\eta_m}{(\eta_m^2-\beta_{mm}^2)}\right.\right.$$

$$\left.-\frac{\frac{3}{2}\frac{\varepsilon_0^2}{\mu^2}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}-\frac{2\frac{k_1}{\mu}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}-\frac{\frac{\varepsilon_0^2}{\mu^2}\sigma_m^*\ \eta_m}{(\eta_m^2-\beta_{mm}^2)}\right]t^4-\frac{(\eta_m^2-\beta_m^2)^2}{5!}\left[\frac{\frac{3}{2}\omega_m^2\frac{\varepsilon_0}{\mu}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}\right.$$

$$+\frac{\frac{9}{2}\frac{k_1\varepsilon_0}{\mu^2}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}+\frac{6\frac{\varepsilon_0^3}{\mu^3}\sigma_m}{(\eta_m^2-\beta_{mm}^2)}+\frac{\sigma_m^*\eta_m\left(\frac{\varepsilon_0^2}{\mu^2}+\frac{k_1}{\mu}\right)}{(\eta_m^2-\beta_{mm}^2)}+\frac{\frac{1}{2}\frac{\varepsilon_0}{\mu}\sigma_m(\eta_m^2+\beta_{mm}^2)}{(\eta_m^2-\beta_{mm}^2)}$$

$$\left.\left.-\frac{2\frac{\varepsilon_0^2}{\mu^2}\sigma_m\beta_{mm}^2}{(\eta_m^2-\beta_{mm}^2)}+\frac{\frac{\varepsilon_0^2}{\mu^2}\sigma_m\eta_m}{(\eta_m^2-\beta_{mm}^2)}\right]t^5+\cdots\right\}\left(\sigma^*\,sinh\frac{\lambda_m}{L}x+sin\frac{\lambda_m}{L}x\right.$$

$$+\sigma_m cosh\frac{\lambda_m}{L}-\sigma_m cos\frac{\lambda_m}{L}x\Big) \tag{83}$$

Following the same procedure as for the case concerning moving mass problem, the lateral responses of the Case II is obtained by solving equations (80) and (81). Hence, the resulting equations are,

$$W_{1mf}(x,t)=\sum_{m=1}^{\infty}\frac{D_1}{\left(\eta_m^2-\omega_{mf}^2\right)}\Bigg\{\frac{\frac{3}{2}\sigma_m\left(\eta_m^2-\omega_{mf}^2\right)}{2!}t^2-\frac{\left(\eta_m^2-\omega_{mf}^2\right)}{3!}\left[\sigma_m^*\eta_m+\eta_m\frac{3}{2}\frac{\varepsilon_0}{2\mu}\sigma_m\right]t^3$$

$$+\frac{(\eta_m^2-\beta_m^2)^2}{4!}\left[\frac{\frac{\sigma_m}{2}(\eta_m^2+\beta_m^2)}{\left(\eta_m^2-\omega_{mf}^2\right)}-\frac{2\sigma_m\beta_{mm}^2}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{\frac{3}{2}\frac{k_1}{\mu}\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{3\frac{\varepsilon_0^2}{\mu^2}\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}\right.$$

$$\left.+\frac{\frac{\varepsilon_0}{\mu}\sigma_m^*\eta_m}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{\frac{\varepsilon_0}{\mu}\eta_m}{\left(\eta_m^2-\omega_{mf}^2\right)}\right]t^4+\frac{(\eta_m^2-\beta_m^2)^2}{5!}\left[\sigma_m^*\eta_m-\frac{\eta_m(\eta_m^2+\beta_m^2)}{\left(\eta_m^2-\omega_{mf}^2\right)}\right.$$

$$-\frac{\beta_m^2\sigma_m^*\eta_m}{\left(\eta_m^2-\omega_{mf}^2\right)}-\frac{\frac{3}{2}\beta_{mm}^2\frac{\varepsilon_0}{\mu}\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{\frac{k_1}{\mu}\eta_m(1+\sigma_m^*)}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{6\frac{k_1\varepsilon_0}{\mu^2}\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}$$

$$+\frac{6\frac{\varepsilon_0^3}{\mu^3}\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}-\frac{\frac{\varepsilon_0^2}{\mu^2}\eta_m(1+\sigma_m^*)}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{\frac{1}{2}\sigma_m\frac{\varepsilon_0}{\mu}\left(\eta_m^2-\omega_{mf}^2\right)}{\left(\eta_m^2-\omega_{mf}^2\right)}-\frac{2\frac{\varepsilon_0}{\mu}\sigma_m\omega_{mf}^2}{\left(\eta_m^2-\omega_{mf}^2\right)}$$

$$\left.-\frac{\frac{\varepsilon_0^2}{\mu^2}\sigma_m^*\eta_m}{\left(\eta_m^2-\omega_{mf}^2\right)}-\frac{\frac{\varepsilon_0^2}{\mu^2}\eta_m}{\left(\eta_m^2-\omega_{mf}^2\right)}\right]t^5+\cdots\Bigg\}\Big(\sigma^* sinh\frac{\lambda_m}{L}x+sinh\frac{\lambda_m}{L}x$$

$$+\sigma_m cosh\frac{\lambda_m}{L}x-\sigma_m cos\frac{\lambda_m}{L}x\Big) \tag{84}$$

$$W_{2mf}(x,t)=\sum_{m=1}^{\infty}\frac{D_1}{\left(\eta_m^2-\omega_{mf}^2\right)}\Bigg\{-\frac{\frac{3}{2}\frac{\varepsilon_0}{\mu}\sigma_m(\eta_m^2-\beta_m^2)}{3!}t^3+\frac{\sigma_m(\eta_m^2-\beta_m^2)^2}{4!}\left[\frac{\frac{\varepsilon_0}{\mu}\eta_m}{\left(\eta_m^2-\omega_{mf}^2\right)}\right.$$

$$\left.-\frac{\frac{3}{2}\frac{\varepsilon_0^2}{\mu^2}\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}-\frac{2\frac{k_1}{\mu}\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}-\frac{\frac{\varepsilon_0^2}{\mu^2}\sigma_m^*\quad\eta_m}{\left(\eta_m^2-\omega_{mf}^2\right)}\right]t^4-\frac{(\eta_m^2-\beta_m^2)^2}{5!}\left[\frac{\frac{3}{2}\omega_m^2\frac{\varepsilon_0}{\mu}\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}\right.$$

$$+\frac{\frac{9}{2}k_1\varepsilon_0\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{6\frac{\varepsilon_0^3}{\mu^2}\sigma_m}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{\sigma_m^*\eta_m\left(\frac{\varepsilon_0^2}{\mu^2}+\frac{k_1}{\mu}\right)}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{\frac{1}{2}\frac{\varepsilon_0}{\mu}\sigma_m\left(\eta_m^2+\omega_{mf}^2\right)}{\left(\eta_m^2-\omega_{mf}^2\right)}$$

$$\left.-\frac{2\frac{\varepsilon_0^2}{\mu^2}\sigma_m\omega_{mf}^2}{\left(\eta_m^2-\omega_{mf}^2\right)}+\frac{\frac{\varepsilon_0^2}{\mu^2}\sigma_m\eta_m}{\left(\eta_m^2-\omega_{mf}^2\right)}\right]t^5+\cdots\Bigg\}\Big(\sigma_m^* sinh\frac{\lambda_m}{L}x+sin\frac{\lambda_m}{L}x$$

$$+\sigma_m cosh\frac{\lambda_m}{L}x-\sigma_m cos\frac{\lambda_m}{L}x\Big) \tag{85}$$

Thus, equations (82) and (83) represent the lateral dynamic deflections of the upper and lower Euler-Bernoulli beams which are elastically restrained at both ends and traversed by a concentrated moving mass, while equations (84) and (85) denote the lateral dynamic deflections of the vibrating beams due to a concentrated moving force.

Numerical Examples

In order to illustrate the procedural technique applied, specific examples are presented in this section. A moving mass of the beam system for which the inertia of the moving load is taken into account has been solved. Also, a moving force problem of the same beam system traversed by a moving load whose inertia effect is neglected has been considered. In the two cases, elastic spring support conditions have been applied to determine the correctness and accuracy of the proposed technique. A computer program, MAPLE 18, have been applied in view of the following values for the parameters of the double-beam system from Ref, (Oniszczuk, 1999).

$$EI = 1.6\times10^4\,Nm^2,\ k_1 = 10Nm^2,\ \mu = 7.5\times10kg\,m^2,\ L = 6m,$$
$$v = 3.3,\ 6.3,\ 9.3, 12.3ms^2;\ \ \varepsilon_0 = 0.10,\ 0.20,\ 0.30,\ 0.35;$$
$$\lambda = 0.25,\ 0.5,\ 0.75,\ 1.0$$

Figure 5.1(i) and 5.1(ii) presents the influence of mass ratio on the transverse deflection of both the upper and lower Euler-Bernoulli beams. The plot indicates that increasing mass ratio lead to a decrease in the response amplitudes of both the upper and lower beams. However, the absolute maximum response amplitude is greater in the case involving lower beam due to moving force.

Figure 5.2(i) and 5.2(ii) represent the effect of variation of velocity on the transverse deflection of the upper Euler-Bernoulli due to moving mass and moving force respectively. It is noticeable that increasing the moving speed of the load caused an increase in the response amplitude of upper beam due to mass and force.

The absolute maximum response amplitude is observed to be greater in the case due to moving force. The same trends are observed in Figure 5.2(ii) and 5.2(iv) for the lower Euler-Bernoulli beam due to moving mass and force.

Effect of variation of viscoelastic parameter $\in_0$ is shown in 5.3(i) and 5.3(iv). The plots in Figures 5.3(i) and 5.3(ii) indicates that increasing $\in_0$, is seen to cause an increase in the response amplitude of upper beam due to moving mass and moving force. It is however noticed that the absolute maximum response amplitude decreases as the load traversed the length of the beam from $x = 0$ *and* $x = L$. The same trend is observed in Figure 5.3(i) and 5.3(iv).

Figures 5.4(i) and 5.4(iv) depicts the effect of stiffness parameter (k_1) on the dynamic response of both the upper and lower Euler-Bernoulli beams. Increasing the value of k_1 in figure 5.4(i) and 5.4(ii) is seen to cause a decrease in the response amplitudes of deflection of the upper Euler-Bernoulli beam due to moving mass and moving force. The same trend is noticed in the case of involving lower beam due to moving force as shown in Figures 5.4(i) and 5.4(iv) respectively.

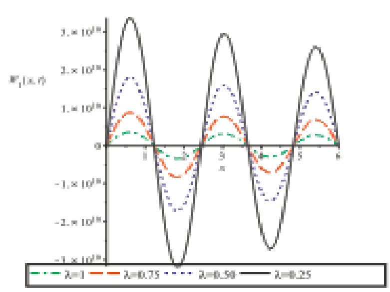

Figure 5.1(i): Graph of deflection against distance varying mass for the upper Euler-Bernoulli beam

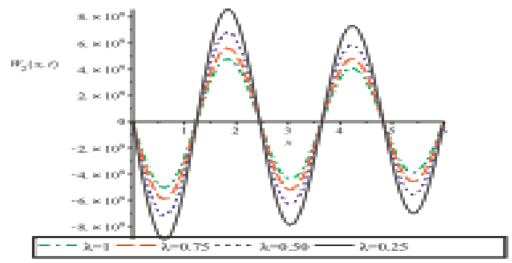

Figure 5.1(ii): Graph of deflection against distance varying mass for the lower Euler-Bernoulli beam

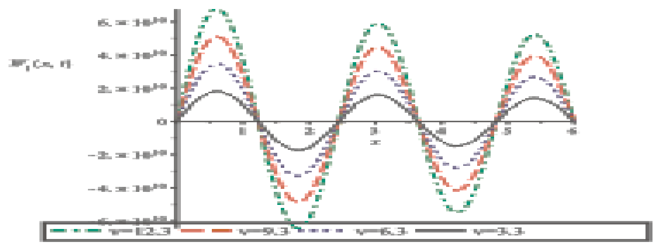

Figure 5.2(i): Graph of Deflection against distance varying velocity for the upper Euler-Bernoulli beam due to moving mass

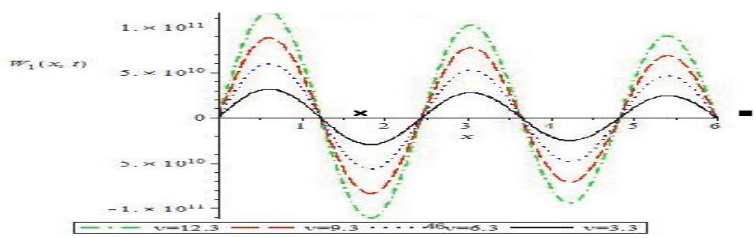

Figure 5.2(ii):Graph of deflection against distance varying velocity for the upper Euler-Bernoulli beam due to moving force

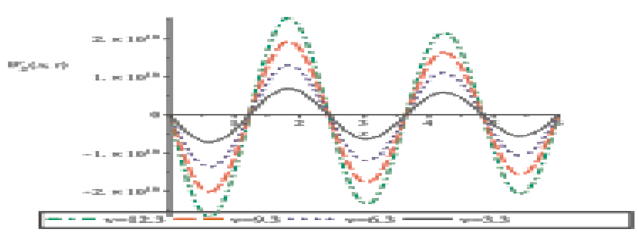

Figure 5.2:(iii): Graph of deflection against distance varying velocity for the lower Euler-Bernoulli beam due to moving mass

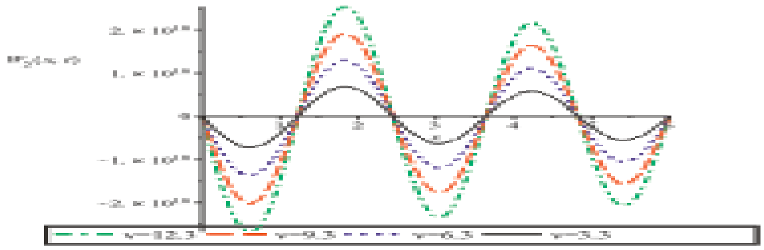

Figure 5.3(iv): Graph of deflection against distance varying velocity for the lower Euler-Bernoulli beam due to moving mass

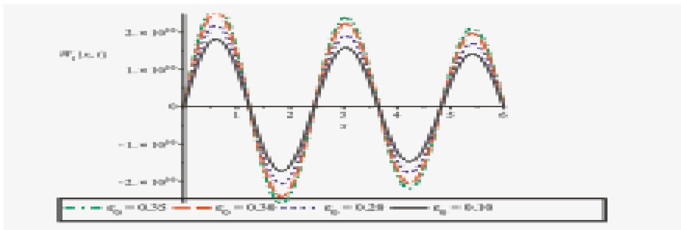

Figure 5.3 (i): Graph of deflection against distance varying damping coefficient for the upper Euler-Bernoulli beam due to moving mass

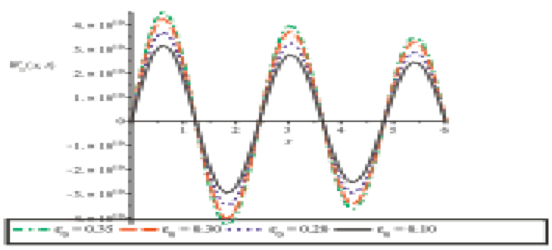

Figure 5.3 (ii): Graph of deflection against distance varying viscoelastic parameter for the upper Euler-Bernoulli beam due to moving force

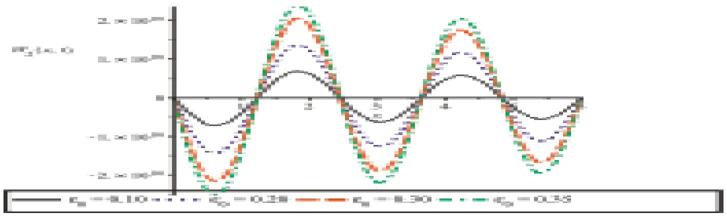

Figure 5.3(iii): Graph of deflection against distance varying damping coefficient for the lower Euler-Bernoulli beam due to moving mass

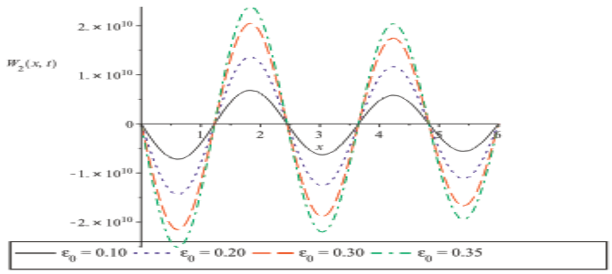

Figure 5.3(iv): Graph of deflection against distance varying damping coefficient for the lower Euler-Bernoulli beam due to moving force

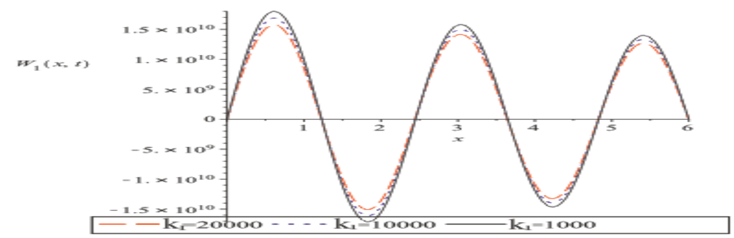

Figure 5.4 (i): Graph of deflection against distance varying stiffness for the

upper Euler-Bernoulli beam due to moving mass

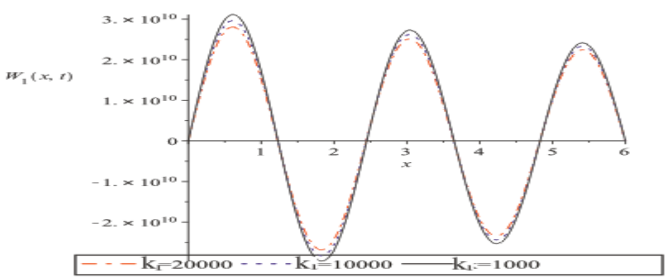

Figure 5.4 (ii): Graph of deflection against distance varying stiffness for the upper Euler-Bernoulli beam due to moving force

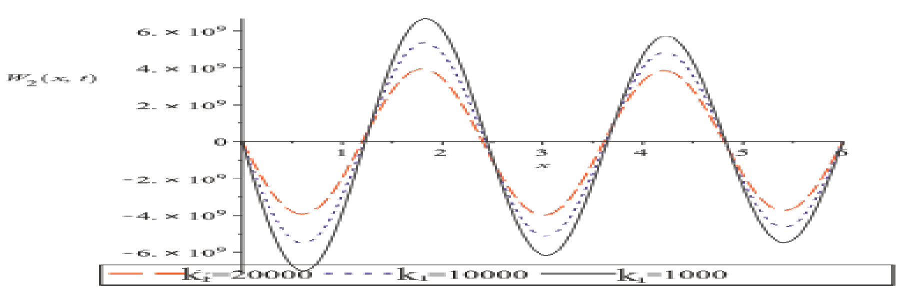

Figure 5.4(iii): Graph of deflection against distance varying stiffness for the lower Euler-Bernoulli beam due to moving mass

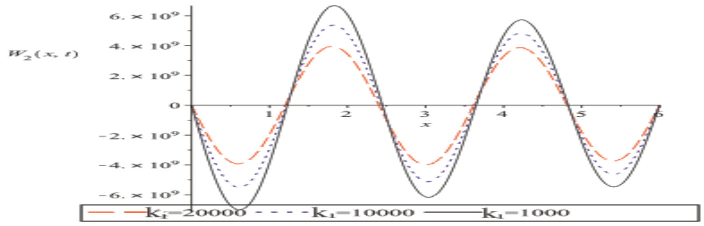

Figure 5.4(iii): Graph of deflection against distance varying stiffness for the lower Euler-Bernoulli beam due to moving mass force

CONCLUSION

The dynamic response of a double beam system whose ends are elastically restrained with Winkler-type interconnected layer under a concentrated moving mass have been investigated. The analysis concerning differential equations of motion characterising the system have been conducted through a solution scheme involving a series variable separable method for the reduction of order of the coupled motion equations, Struble's asymptotic method for the decoupling of the coupled equations and differential transform method for the simplification of the resulting decoupled second order ordinary differential equations of motion. One of the limitations of the proposed method is that, it is a small parameter method and it is known to be valid in small region. However, the enlargement of the convergence domain could be achieved through "After Treatment Technique" (30). Another limitation is the associated variational equation iteration. The influence of the mass on the deflection of the beams is observed to cause a decrease in the response amplitudes concerning both beams. However, the absolute maximum response amplitude is greater in the case involving lower beam due to a moving force. Increasing the stiffness parameter on the dynamic response of the beams is also seen to cause a decrease in the response amplitude due to both moving mass and moving force. The effect is lower in the case due to a moving mass.

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