FORCED RESPONSE VIBRATION OF SIMPLY SUPPORTED BEAMS WITH AN ELASTIC PASTERNAK FOUNDATION UNDER A DISTRIBUTED MOVING LOAD

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ABSTRACT

In this study, the response of two homogeneous parallel beams with two-parameter Pasternak elastic foundation subjected to a constant uniform partially distributed moving force is considered. On the basis of Euler-Bernoulli beam theory, the fourth order partial differential equations of motion describing the behavior of the beams when subjected to a moving force were formulated. In order to solve the resulting initial-boundary value problem, finite Fourier sine integral technique and differential transform scheme were employed to obtain the analytical solution. The dynamic responses of the two beams obtained was investigated under moving force conditions using MATLAB. The effects of speed of the moving force, layer parameters such as stiffness (K_0) and shear modulus (G_0) have been conducted for the moving force.Various values of speed of the moving load, stiffness parameters and shear modulus were considered. The results obtained indicates that response amplitudes of both the upper and lower beams increases with increase in the speed of the moving load. Increasing the stiffness parameter is observed to cause a decrease in the response amplitudes of the beams. The response amplitudes decreases with increase in the shear modulus of the linear elastic layer.

Keywords: Dynamic response, Two-parameter Pasternak elastic Foundation, Euler-Bernoulli beam, Moving Force, Foundation Parameter.

INTRODUCTION

This work is concerned with the study of elastic beams. Beams used in various mechanical systems are subjected to forces, which cause them to deform. The dynamic behaviour of beam-type structures on elastic foundations under the influence of moving loads has been subject of concern to numerous researchers in the field of mechanical and structural engineering. Sun (2001), Sun and Deng (1998)].Fryba(1999), in particular presented a detailed solution techniques to problems of moving loads on Euler-Bernoulli beam supported with one-parameter foundation model. The analysis of such structures supported by elastic foundations traversed by moving loads, mostly considered the relatively simple model of Winkler (1867) which consists of linear independent layer of closely spaced elastic springs. The constant of proportionality of these springs is termed modulus of subgrade reaction. The Winkler's model which is also termed one-parameter models [Eisemberger and Claslomik (1987) could not adequately represent the characteristics of foundation materials in engineering applications since it assumes no interaction between the lateral springs. In an attempt to eliminate the shortcoming attributed to one-parameter foundation model, an improved theory called a two-parameter foundation model was proposed by Pasternak (1954) for the analysis of the dynamic behavior of beams under moving loads. This model has been considered to find a physically close and mathematically simple foundation model to represent foundation layer. The two-parameter Pasternak model was achieved when the ends of the vertical springs are connected with an incompressible vertical element of a beam, which deforms only by transverse shear. The two parameters of the foundation layer are the stiffness of the springs and the shear rigidity of the beam. Among other proposed elastic foundation models such as; Filonenko-Borodich (1940), Hetenyi (1946), Kerr (1964), the most natural extension of the one-parameter elastic foundation model is the two-parameter elastic foundation model of Pasternak, with shear modulus as second parameter.

The dynamic analysis of beam-type structures supported by two-parameters foundation models under moving loads with uniform velocity has been investigated.The reactive measure of the elastic layer under the action of a distributed load as described by Kerr (1964) was introduced in the formulation of the differential equation of motion. The results obtained indicated that a two-parameter elastic foundation model is a more realistic representation of foundation models on the basis of practical considerations particularly for rocks or gravelly soils.

Forced transverse vibration & analysis of a simply supported and elastically connected double-beam system was conducted by Oniszczuk (2003). The system was subjected to an arbitrary distributed moving load. The method of classical modal expansion was applied to determine the dynamic responses of the beams due to the harmonic forces of excitation. Lancu*et al*. (2006) applied a finite element method to investigate the bending behaviour of beams resting on two-parameter elastic foundation. Li and Hua (2007) investigated the vibration of two beams. The beams were elastically connected under different boundary supports. The method of spatial finite element was utilized to obtain some numerical solutions required for the natural frequencies.

A great number of the analysis conducted employed the method of integral Fourier series transformation to solve differential equations of motion of beams under the influence of forced vibration [Fryba (1999), Mallik *et al*. (2006), Yong and Yang (2008), Rajib U.I (2012)]. On the basis of Rayleigh beam theory, forced response vibration analysis of double beam system supported by a two-parameter foundation with compressive axial loading was developed by Mohammadi and Nasirshoaibi(2015). The governing fourth order partial differential equations describing the motion of the beams were formulated and solved using variable separable method and classical modal expansion method. The resultant dynamic vibration developed in response to forces induced by harmonic excitation was discussed. The condition of resonance was developed and analyzed. This is to prevent the possibility that a resonance condition would occur which can cause a sudden catastrophic failure of mechanical or structural element.

The analysis of the forced response vibration of a simply supported double Euler-Bernoulli beam system which is elastically connected by Pasternak middle layer under a uniform distributed moving force is studied in this paper.

MATHEMATICAL FORMULATION OF THE PROBLEM

The elastically connected beams subjected to a moving force shown in Figure 1. The response of the beams to a moving force is the object of investigation. The study includes determining the deflections of two beams when subjected to a moving force. The formulation of the governing equation for the two elastically connected beams, assumed that the mass of each of the beams is negligible when compared with the mass of the force. The force considered here is in the form of a moving force of constant magnitude.

The force is a uniform partially distributed force. The behavior of the beam material is linearly elastic and the cross section is identical through the length x=0 to x=L of the beam whose plane of geometry is one. The cross-section shear modulus is not negligible while ignoring elastic axial deformations. Also the axial forces F_0 acting at the beam ends does not vary with time. It should also be noted that the two beams are undamped identical having the same length L, and mass per unit length μ.

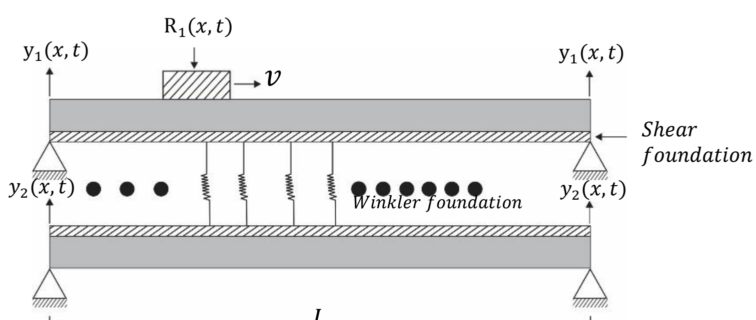


Figure 1: *A double-beam system subjected to a moving force. [Abu-Hilal (2006)]*

The beams has been modeled as two-parameter Pasternak model and was subjected to a distributed moving force. According to the Euler-Bernoulli beam theory, the dynamic responses $y_1(x,t)$ of the upper beam and $y_2(x,t)$ of the lower beam satisfies the following pair of fourth order governing partial differential equations:

$$EI\frac{\partial^4 y_1(x,t)}{\partial x^4}+\mu\frac{\partial^2 y_1(x,t)}{\partial t^2}-K_0[y_1(x,t)-y_2(x,t)]$$
$$-G_0\left[\frac{\partial^2 y_1(x,t)}{\partial x^2}-\frac{\partial^2 y_2(x,t)}{\partial x^2}\right]=R_1(x,t) \quad (1)$$

and

$$EI\frac{\partial^4 y_2(x,t)}{\partial x^4}+\mu\frac{\partial^2 y_2(x,t)}{\partial t^2}-K_0[y_2(x,t)-y_1(x,t)]$$
$$-G_0\left[\frac{\partial^2 y_2(x,t)}{\partial x^2}-\frac{\partial^2 y_1(x,t)}{\partial x^2}\right]=0 \quad (2)$$

where $R_1(x,t)$ is the applied force defined as:

$$R_1(x,t)=\begin{cases}-\frac{P}{\varepsilon}\left[H\left(x-\xi+\frac{\varepsilon}{2}\right)-H\left(x-\xi-\frac{\varepsilon}{2}\right)\right]; & \xi=vt+\frac{\varepsilon}{2}\\ 0,\ otherwise & \end{cases} \quad (3)$$

Theboundary conditions associated with equations (1) and (2) are:

$$y_1(0,t)=0=y_1(L,t);\quad \frac{\partial^2 y_1(0,t)}{\partial x^2}=0=\frac{\partial^2 y_1(L,t)}{\partial x^2} \quad (4)$$

$$y_1(0,t)=0=y_1(L,t);\quad \frac{\partial^2 y_2(0,t)}{\partial x^2}=0=\frac{\partial^2 y_2(L,t)}{\partial x^2} \quad (5)$$

and the corresponding initial conditions are:

$$y_1(x,0)=0=\frac{\partial y_1(x,0)}{\partial t};\ y_2(x,0)=0=\frac{\partial y_2(x,0)}{\partial t} \quad (6)$$

The symbols and parameters used in equations (1)-(2) have the following meanings unless otherwise redefined in the subsequent discussions.

E -Young modulus of elasticity, I -Cross-sectional moment of inertial

EI - Flexural rigidity of the beam,R_1-Uniform partially distributed force of constant magnitude μ - Constant mass per unit length of the beam, K_0 - Stiffness parameter

G_0- Shear modulus that account for the shear interaction among the springs

L - Fixed length of the beam, H - Heaviside function such that $\delta(x)=H^1(x)$ and $\delta(x-vt)$is the Dirac delta functions at point $x=vt$., x - Axial coordinate, v - Velocity of the moving force, t - Time, $\varepsilon-$ Fixed length of the beam

Thus, the initial boundary-value problem to be analysed is described in equations (1) - (6)

Method of Solution

In order to solve the initial boundary-value problem described in equations (1)-(6), integral transformation was first introduced. This method is adopted since it has been proved suitable in the study of moving loads [Gbadeyan and Oni (1995), Rajib UI, et al. (2012)]; Mohammadi and Nasirshoaibi (2015)]. The second stage of solution was achieved by using differential transform method which has equally been demonstrated to be efficient in the solution of both linear and non-linear partial and ordinary differential equations (Zhon 1986; Allennejad el al. 2009; Raslan et al 2012; Gbadeyan and Hammed, 2017).

Therefore, the initial boundary value problem described in equations (1)-(2) is solved by assuming the following finite Fourier sine integral transformation in the equations.

$$\bar{y}_m(n,t) = \int_0^L y_m(x,t)\,sin\frac{n\pi x}{L}dx\,; \quad m = 1,2\,;\; n = 1,2,\ldots \tag{7}$$

and the corresponding inverse is of the form

$$y_m(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\bar{y}_m(n,t)\,sin\frac{n\pi x}{L}, \quad m = 1,2. \tag{8}$$

In view of equations (4)-(6), equation (7) is applied to equation (1) and (2) respectively. After some simplifications, the resulting equations are:

$$\ddot{\bar{y}}_1(n,t) + \omega_n^2\bar{y}_1(n,t) + \alpha_n\bar{y}_1(n,t) + \beta_n\bar{y}_2(n,t) = \frac{Mg}{\mu}sin\frac{n\pi\upsilon t}{L} \tag{9}$$

and

$$\ddot{\bar{y}}_2(n,t) + \omega_n^2\bar{y}_2(n,t) + \alpha_n\bar{y}_2(n,t) + \beta_n\bar{y}_1(n,t) = 0 \tag{10}$$

where

$$\alpha_n = \frac{1}{\mu}\left[\frac{n^2\pi^2}{L^2}\mathrm{G}_0 - \mathrm{K}_0\right] \text{ and } \beta_n = \frac{1}{\mu}\left(\mathrm{K}_0 - \mathrm{G}_0\frac{n^2\pi^2}{L^2}\right) \tag{10a}$$

$\omega_n^2 = \frac{EI}{\mu}\frac{n^4\pi^4}{L^4}$such that ω_n is the natural frequency of the beam.

Thus, the governing fourth order partial differential equations (1) and (2) have thereby, been reduced to the second order ordinary differential equations (9) and (10) using finite Fourier sine transformation.

DIFFERENTIAL TRANSFORM METHOD (DTM)

The reduced governing differential equations of beam motion (9) and (10) are solved using DTM. The concept of DTM which was introduced by Zhou (1986) to solve intial boundary-value problems in enginnering applications have been applied by a great number of researchers to solve a wide range of moving load vibration-induced problems concerning mechanical systems [Ho and Chen (1998), Atternejad and Shahba (2008), Gbadeyan and Agboola (2012), Raslan *et al.* (2012), Gbadeyan and Hammed (2017)]. The basic idea involves considering an analytic function $\bar{y}_m(n,t)$ having a continuous deivatives within the considered domain such that

$$\mathrm{Y}_m(k) = \frac{1}{k!}\left[\frac{d^k\bar{y}_m(n,t)}{dt^k}\right]_{x=x_0} \tag{11}$$

where $\bar{y}_m(n,t)$ is the original function and $Y_m(k)$ is the transformed function. The differential inverse transform of $Y_m(k)$is defined as

$$\bar{y}_m(n,t) = \sum_{k=0}^{\infty} Y_m(k)(t-t_0)^k \tag{12}$$

Considering equations (11) and (12), the resulting equation is

$$\bar{y}_m(n,t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \frac{d^k Y_m(k)}{dt^k}\bigg|_{t=0} \tag{13}$$

When the values of $t_0 = 0$, equation (13) yields

$$\bar{y}_m(n,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{d^k Y_m(k)}{dt^k}\right]_{t=0} \tag{14}$$

Hence,

$$\bar{y}_m(n,t) = \sum_{k=0}^{\infty} Y_m(k) t^k \tag{15}$$

The main difference between Taylor series method and differential transform method is that the former requires computations of higher order derivatives that are quite formidable while the latter involves iterative procedure instead. In real life application similar to the present situation, the function $\bar{y}_m(n,t)$ should be a finite series such that equation (15) becomes

$$\bar{y}_m(n,t) = \sum_{k=0}^{N} Y_m(k) t^k \tag{16}$$

Hence, $\sum_{k=N+1}^{\infty} Y_m(k) t^k$ is regarded as negligibly small such that the values of N is decided by the convergence of natural frequency in this study. The fundamental operations that are frequently used in the transformation of the equation of motion and the boundary conditions are listed in the Tables 1 and 2.

Table 1: Basic Theorem of Differential Transform Method for Equations of motion

Original function	**Transformed function**
$y(t) = u(t) + v(t)$	$Y(k) = \bar{U}(k) + \bar{V}(k)$
$y(t) = cu(t)$	$Y(k) = c\bar{U}(k)$
$y(t) = \frac{du(t)}{dt}$	$Y(k) = (k+1)\bar{U}(k+1)$
$y(t) = \frac{d^n u(t)}{dt^n}$	$Y(k) = (k+1)(k+2)\dots(k+n-1)(k+n)\bar{U}(k+n)$
$y(t) = sin\ at$	$Y(k) = \frac{1}{k!}a^k\ sin\left(\frac{k\pi}{2}\right)$
$y(t) = \cos at$	$Y(k) = \frac{1}{k!}a^k\ cos\left(\frac{k\pi}{2}\right)$
$y(t) = sinh\ at$	$Y(k) = \frac{1}{2k!}[(a)^k - (-a)^k]$
$y(t) = cosh\ at$	$Y(k) = \frac{1}{2k!}[(a)^k + (-a)^k]$

Table 2: Theorem of Differential Transform Method for Boundary Conditions

Original BC $(x = 0)$	T-BC $(x = 0)$	Original BC $(x = 0)$	T-BC $(x = 0)$
$w(0) = 0$	$\bar{W}(0)$	$w(1) = 0$	$\sum_{k=0}^{\infty} \bar{W}(k) = 0$
$\frac{dw(0)}{dx} = 0$	$\bar{W}(1)$	$\frac{dw(1)}{dx} = 0$	$\sum_{k=0}^{\infty} \bar{W}(k) = 0$
$\frac{d^2w(0)}{dx^2} = 0$	$\bar{W}(2)$	$\frac{d^2w(1)}{dx^2} = 0$	$\sum_{k=0}^{\infty} k(k-1)\,\bar{W}(k) = 0$
$\frac{d^3w(0)}{dx^3} = 0$	$\bar{W}(3)$	$\frac{d^3w(1)}{dx^3} = 0$	$\sum_{k=0}^{\infty} k(k-1)\,(k-2)\bar{W}(k) = 0$

In view of equation (16), and with the application of the results in Tables 1 and 2 to equations (9) and (10), the unknown functions $\bar{y}_m(n,t)$for $m = 1, 2$ are obtained as

$$\bar{y}_1(n,t) = \left(\frac{Mg}{\mu}\right)\left(\frac{n\pi v}{L}\right)\left[\frac{1}{3!}t^3 - \frac{1}{5!}\left[\left[\left(\frac{n\pi v}{L}\right)^2 + \omega_n^2\right] + \alpha_n\right]t^5\right.$$
$$\left. + \frac{1}{7!}\left[\left[\left(\frac{n\pi v}{L}\right)^4 + \omega_n^2\left[\left[\left(\frac{n\pi v}{L}\right)^2 + \omega_n^2\right] + \alpha_n\right] + \beta_n^2\right]t^7 + \cdots\right] \quad (17)$$

$$\bar{y}_2(n,t) = -\beta_n\left(\frac{Mg}{L}\right)\left(\frac{n\pi v}{L}\right)\left[\frac{1}{5!}t^5 + \frac{1}{7!}\left[\omega_n^2 + 2\alpha_n + \left[\left(\frac{n\pi v}{L}\right)^2 + \omega_n^2\right]\right]t^7 + \cdots\right] \tag{18}$$

On substituting equations (17) and (18) into equation (8) for the case $m = 1$ and $m = 2$ respectively, the resulting equations are

$$y_1(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\bar{y}_1(n,t)\,sin\frac{n\pi x}{L} \tag{19}$$

$$y_2(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\bar{y}_2(n,t)\,sin\frac{n\pi x}{L} \tag{20}$$

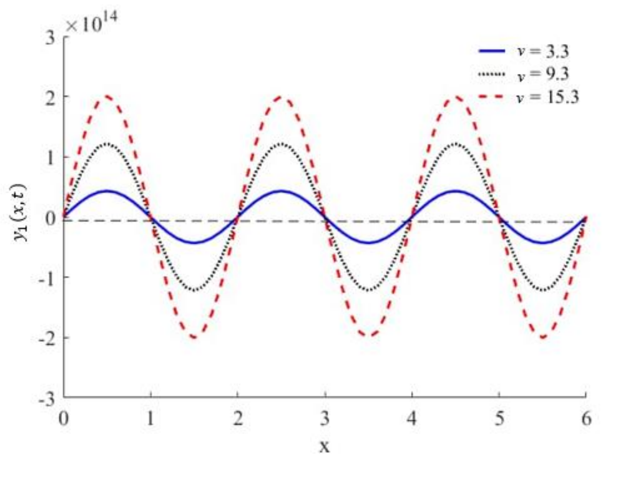
Hence, $y_1(x,t)$ and $y_2(x,t)$ represent the dynamic responses of the simply supported upper and lower Euler-Bernoulli beams with Pasternak elastic middle layer under a uniform partially distributed moving force $R_1(x,t)$.

RESULTS AND DISCUSSION

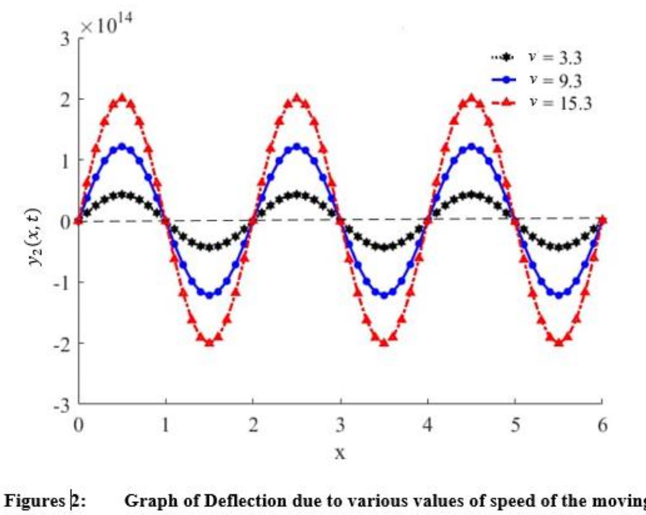
Numerical analysis of the results obtained in equations (19) and (20) is conducted. This has to do with situations when the elastically connected beams under a partially distributed moving force is assumed undamped. The rotatory inertia effects was ignored while those of shear modulus was taken into consideration. Also, the mass of the beams is assumed negligible when compared to that of the moving force.

In order to obtain the layer shear stiffness effects as well as those of other interacting beam parameters, the analytic results obtained represented by equations (19) and (20) were simulated using MATLAB. These numerical computation was achieved for the two beams by making use of the following values: [Abu-Hilal (2006)] for the purpose of comparison. $\mu = 0.075$; $EI = 16{,}000$; $g = 10$; $L = 6$; $\varepsilon_0 = 0.10, 0.20, 0.30, 0.35$; $x = 3$; $k = 10$; $\pi = \frac{22}{7}$; $t = 0.5$

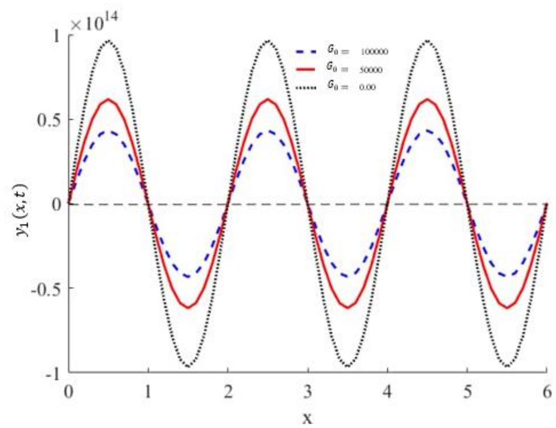
5. SIMPLY SUPPORTED MOVING FORCE GRAPHS



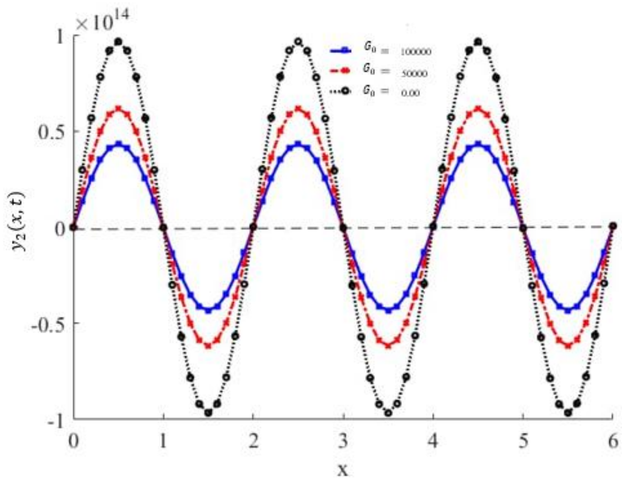
Figures 1: Graph of Deflection due to various values of speed of the moving Force on the upper beam



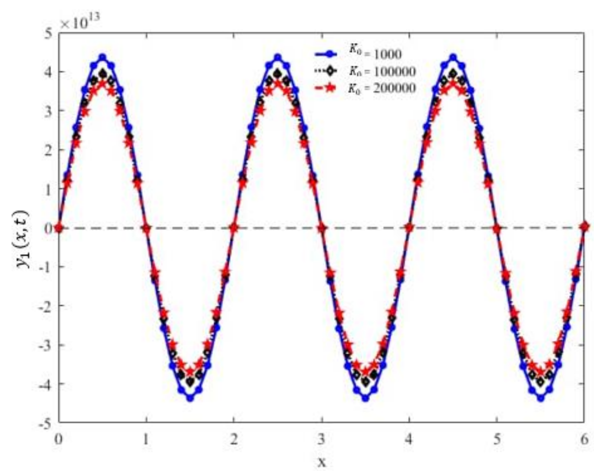
Figures 2: Graph of Deflection due to various values of speed of the moving force on the lower beam



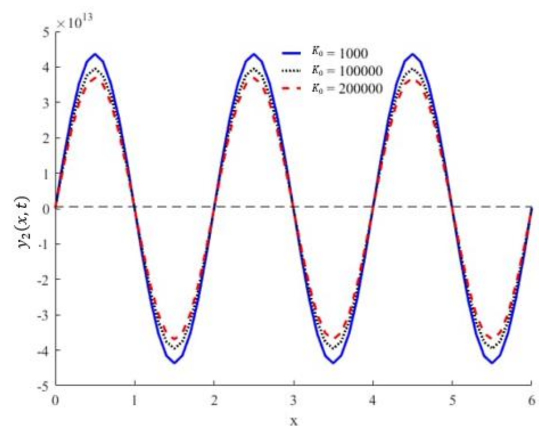
Figures 3: Graph of Deflection of the upper beam due to variation of shear modulus



Figures 4: Graph of Deflection of the lower beam due to variation of shear modulus



Figures 5: Graph of Deflection of the upper beam due to variation of stiffness parameter



Figures 6: Graph of Deflection of the lower beam due to variation of stiffness parameter

Figures 1 and 2 are the respective plots of variation of velocity of the load on the transverse deflection of the upper and lower beams due to moving force. An increase in velocity is observed to cause an increase in the response amplitude of both the upper and lower beams.

Figures 3 and 4 shows the variation due to shear modulus G_0 of the foundation layer. Increasing the shear modulus is seen to cause a decrease in the absolute response amplitude of both the upper and lower beams. However, setting the value of shear modulus to zero, the same results as those of Abu-Hilal (2006) was obtained.

Figures 5 and 6 shows the absolute response amplitudes due to variation of stiffness parameter K_0 due to the moving force on the dynamic response of the beams. Increasing the values of K_0 is seen to cause a decrease in the response amplitudes of the beams. The absolute response amplitudes is observed to be greater in the upper beam when compared with those observed in the lower beam.

CONCLUSION

This paper examines the dynamics responses of a double Euler-Bernoulli beam system which is elastically connected by a two-parameter Pasternak foundation model under the action of a moving distributed force. Finite Fourier sine transformation was employed to reduce the fourth order partial differential equations describing motion of the beams to second order ordinary differential equations. The dynamic response of the beams were then obtained using differential transformation method. Numerical computations were conducted to analyze the dynamic responses obtained for the beams supported by Pasternak foundation, with different values of the foundation interacting parameters including various value of speed of the moving force.

The method employed has been found efficient in the solution of governing equations of beam motion supported by two-parameter Pasternak foundation under a moving force. The speed of the moving force affects the dynamic response of the beams.

Shear and stiffness parameters have a significant effect on the beam responses. It is observed that increasing these foundation parameters causes a decrease in the absolute response amplitudes of the beams. This study can be extended to a moving mass problem of double beam systems where axial forces are taken into account.

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