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DYNAMICS OF A SIMPLY SUPPORTED TWO PARALLEL BEAMS WITH VARIABLE ELASTIC INTERCONNECTED LAYER UNDER A MOVING MASS

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ABSTRACT

In the paper, dynamics of two parallel beams which is interconnected through a nonlinear variable viscoelastic Winkler-type layer under a moving concentrated load is investigated. The elastic characteristics of a simply supported Euler-Bernoulli beam system have been applied. The motion characteristics due to force of excitation on the system is described by a coupled governing fourth order partial differential equations of motion with variable coefficient. A solution scheme involving seperable method have been applied to obtain a coupled coupled second order ordinary differential equations. The decoupling and simplification of the coupled system yields an uncoupled sets of second order ordinary differential equations and this has been attained using an asymptotic method of Struble. A semi analytic differential transform method is applied to solve the resulting equations. A numerical experiment is explored to demonstrate the simplicity and efficiency of the method employed. The effects of various parameters including speed of the moving load, inertia of the moving load, stiffness and viscoelastic parameters of the interconnected layer were obtained. The result indicates that increasing the load mass has caused a decrease in the response amplitude of deflection of the two beams and this is found to be in good agreement with the existing results.

Keywords: Parallel beams, Moving concentrated load, Winkler-type layer, Euler-Bernoulli beam, Force of excitation.

INTRODUCTION

The problem of concerning the assessment of the dynamic behavior of elastic structures under moving loads in engineering applications have been of great concern to various researchers in engineering, applied mathematics and physics disciplines. In most of the previous studies concerted efforts have been deployed at studying free and forced vibration of uniform single Euler-Bernoulli beam structures due to their various significance in engineering practices. In the excellent monograph of Fryba (1972), a comprehensive analysis of techniques for the analysis of the problems concerning vibrating structures under moving loads were developed. Oniszczuk (2000) presented the free vibrations of two parallel simply supported beam which are continuously connected through a Winkler-type elastic layer. The differential equations of motion is formulated and solved using the method of classical Euler-Bernoulli beam theory. Li et al. (2016) investigated the dynamical behaviour of a double beam system which are interconnected by a viscoelastic layer. A semi-analytical method has been developed to analyze the natural frequencies and corresponding mode shapes of the vibrating configuration. The iterated modal-expansion method is further applied to determine reponses due to forced vibration of the vibration.

In the analysis conducted by Mirzabeigy et al. (2016), free transverse vibration of the two parallel beams which are interconnected through variable stiffness Winkler-type elastic layer is considered. The system has been characteristics by Euler-Bernoulli beam system with the translational and rotational elastic end supports. The solution of the motion equation was achieved through a semi analytical approach called differential transform method. The result obtained are relatively new and to compare with the existing contributions is scarce and hence the justification is provided for in special cases presented. Seelig and Hoppmann (1964) investigated the free vibration of two parallel beams which are elastically interconnected through the development and solution of the partial differential equations of motion.

Abu-Hilal (2006) studied the dynamic response of a double Euler-Bernoulli beam system traversed by a constant moving load. A detail investigation on the effects of a number of parameters including moving speed of the load, viscoelastic layer on the dynamic response of the beams was investigated. Zhang et al (2008) investigated vibration of buckling of double-beam system under compressive axial loading. The effect of compressive axial load on forced transverse vibration of a double –beam was studied by Zhang et al. (2008). Impact on an elastically connected double-beam system study was conducted by Seeling and Hoppmann (2011). Stojanovic et al (2011) also studied the effect of rotary inertia shear on vibration and buckling of a double beam system under compressive axial loading. Stojanovic and Kozic

(2012) carried out an extensive investigation into forced vibration and buckling of a Rayleigh and Timoshenko double-beam system interconnected by a Winkler elastic layer under compressive axial loading. The dynamic behavior of a double Rayleigh beam system traversed by uniformly partially distributed moving load was studied by Gbadeyan and Agboola (2012).

In the investigation by Kelly and Nicely (2014), an exact solution for free vibrations of a series of uniform Euler-Bernoulli beams connected by viscoelastic layer developed.The forced vibration problem of the system is analyzed using finite Fourier and Laplace integral transformations.

Mohammedi and Nasirshoaibi (2015), investigated the forced traverse vibration of an elastically connected simply-supported double Rayleigh Beam system with a Pasternak middle layer subjected to compressive axial load. Mirzabeigy and Madoliat (2015) presented free transverse vibration of two elastically restrained beams which are partially connected by an elastic Winkler layer. The natural frequencies were derived by deploying a semi-analytical differential transform method.

Free transverse vibration analysis of two parallel beams interconnected through a variable stiffness Winkler-type layer was presented by Mirzabeigy et al (2016). A semi-analytical solution of the free vibration was achieved using differential transform method. Lie et al (2016) also presented a semi-analytical method to investigate the natural frequencies and mode shape of a double-beam system interconnected by viscoelastic layer. The modal expansion method iterated method was applied to determine the forced vibration responses in the double-beam based on the natural frequencies and mode shapes obtained from the free-vibration analysis.

A new modified multi-level residue harmonic balance method was applied by Rahman and Lee (2017), to investigate the forced nonlinear vibration of axially loaded double beams. The main advantage of the method was that a set of decoupled nonlinear algebraic equations has been generated at each solution level. This has invariably reduces the computational efforts compared with solving the coupled nonlinear algebraic equations generated in the classical harmonic balance.

Mirzabeigy and Madoliat (2019) presented a note on free vibration of double beam system of small amplitude with interconnected nonlinear dynamic layer. The effect of interconnected layer was investigated by first making a distinction between synchronous and asynchronous motion of beams. The results obtained shows that the connecting layer mainly affected the first mode asynchronous frequency, having little effect on higher mode frequency.

To the best knowledge of the authors of this article, most of the earlier works concerning vibration of two beams interconnected through a nonlinear variable Winkler-type layer along the length of beams under moving loads have been limited to those acted upon only by moving forces. The implication is that the effects of the inertial of the moving loads have not been accounted for in Mirzabeigy et al. (2016). In this study, vibration analysis of two parallel beams interconnected through a nonlinear variable stiffness under a moving load whose inertial is non-negligible is conducted using a solution scheme involving separable method, asymptotic method of Struble and Differential Transform method (DTM).

MATHEMATICAL FORMULATION OF THE PROBLEM

A system which is composed of two undamped, finite. prismatic and homogenous parallel beams as shown in Figure 1, is considered. The beams are simply supported at both ends. The continuous interconnected nonlinear viscoelastic Winkler-inner layer which is a function of spatial coordinate x, stiffness c and a damping parameter ε_0 is assumed. It is also assumed that both beams have the same length L and mass per unit length μ.

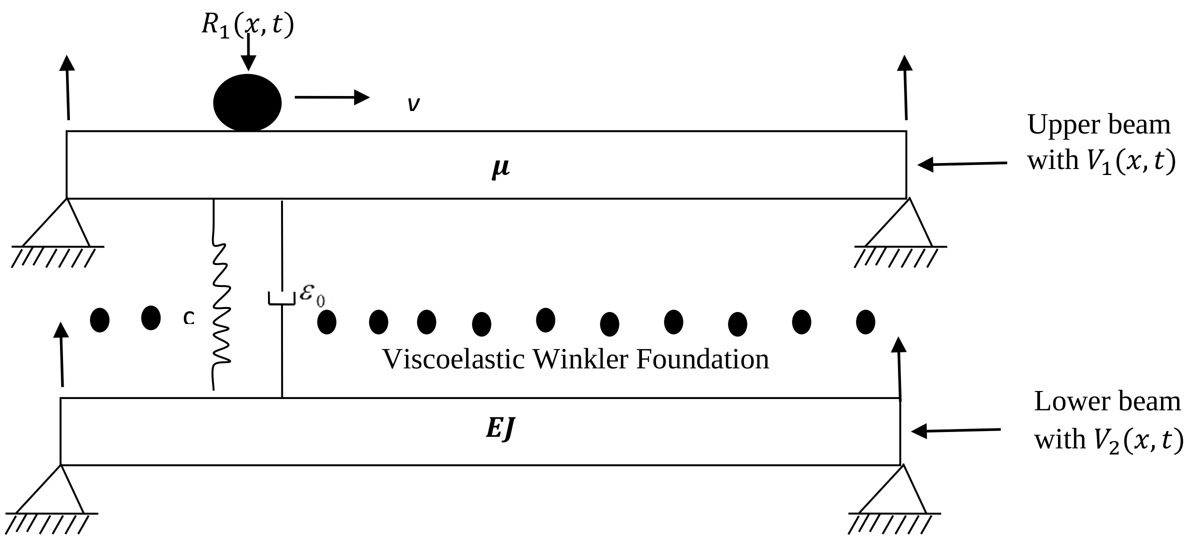

Figure 1: A Double Euler-Bernoulli beam System

The beam's modulus of elasticity is E while the cross sectional moment of inertia is J. The constant gravitational force is g ($9.81ms^2$) and t is time. The upper beam is under the constant influence of an excitation load $R_1(x,t)$ of mass M_L moving with a steady speed v along the length of the beams.

On the basis concerning Euler-Bernoulli beam theory, the dynamic vibration responses $V_1(x,t)$ of the upper beam and $V_2(x,t)$ of the lower beam fulfilled the following pair of coupled governing partial differential equations of motion.

$$EJ\frac{\partial^4 V_1(x,t)}{\partial x^4} + \mu\frac{\partial^2 V_1(x,t)}{\partial t^2} + \left[f(x) + \varepsilon_0\frac{\partial}{\partial t}\right][V_1(x,t) - V_2(x,t)] = R_1(x,t) \quad (1)$$

$$EJ\frac{\partial^4 V_2(x,t)}{\partial x^4} + \mu\frac{\partial^2 V_2(x,t)}{\partial t^2} + \left[f(x) + \varepsilon_0\frac{\partial}{\partial t}\right][V_2(x,t) - V_1(x,t)] = 0 \quad (2)$$

The dynamic load function is,

$$R_1(x,t) = M_L\left[g - \left(\frac{\partial^2 V_1(x,t)}{\partial t^2} + 2v\frac{\partial^2 V_1(x,t)}{\partial x\partial t} + v^2\frac{\partial^2 V_1(x,t)}{\partial x^2}\right)\right]\delta(x - vt) \quad (3)$$

$$f(x) = c(1 + c_1x + c_2x^2) \quad (4)$$

where $c,\ c_1,\ c_2$ are constants, $0 \le x \le L$.

The function $\delta(\cdot)$ is the Dirac delta function usually expressed as

$$\delta(\cdot) = \begin{cases} 0, & x \ne 0 \\ \infty, & x = 0 \end{cases} \quad (5)$$

We also note that the Dirac delta function $\delta(x - vt)$ is an even function. Hence, it is expressed as a Fourier cosine series [1,3],

$$\delta(x - vt) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L}\cos\frac{n\pi}{L}x \quad (6)$$

Since the dynamic beams is simply supported, the following elastic conditions are defined [25, Li, *et al* (2016), Mirzabeigy *et. al.* (2016)]:

$$V_1(o,t) = 0 = V_1(L,t) \quad (7)$$

$$V_2(o,t) = 0 = V_2(L,t) \quad (8)$$

$$\frac{\partial^2 V_1(o,t)}{\partial x^2} = 0 = \frac{\partial^2 V_1(L,t)}{\partial x^2} \quad (9)$$

$$\frac{\partial^2 V_2(o,t)}{\partial x^2} = 0 = \frac{\partial^2 V_2(L,t)}{\partial x^2} \quad (10)$$

The corresponding initial conditions at $t = 0$ are

$$V_1(x,t) = 0 = \frac{\partial V_1(x,t)}{\partial t} \quad (11)$$

$$V_2(x,t) = 0 = \frac{\partial V_2(x,t)}{\partial t} \quad (12)$$

METHOD OF SOLUTION

The initial-boundary-value problem as defined in equations (1), (2) and (3) are solved by introducing a set of solution scheme which involves; (i) a series variable separable method, to reduce the coupled fourth order partial differential equations of motion to a set of coupled second order ordinary differential equations. (ii) the modification of asymptotic method of Struble to simplify the resulting equations in step (i). (iii) solving the simplified set of equations in step (ii) by applying a semi-analytical scheme of differential transformation initially conceived by Zhou (1986).

THE TRANSFORMED EQUATIONS

In order to solve the coupled fourth order partial differential equations (1) and (2), the following series are assumed for equations (1) and (2),

$$V_1(x,t) = \sum_{m=1}^{\infty} \alpha_m(m,t) U_m(x) \quad (13)$$

$$V_2(x,t) = \sum_{m=1}^{\infty} \beta_m(m,t) U_m(x) \quad (14)$$

where $\alpha_m(m;t)$ and $\beta_m(m;t)$ are the unknown functions of time to be determined later. $U_m(x) = sin\frac{m\pi}{L}x$ is the known mode shape function of a simply supported single beam [Michaltsas (1996), Oniszeuk (2000), Oniszeuk (2003)]

Introducing equations (13) and (14) on equations (1) and (2), the obtained equations are,

$$EJ\sum_{m=1}^{\infty} \alpha_m(m,t)U_m^{iv}(x) + \mu\sum_{m=1}^{\infty} \ddot{\alpha}_m(x,t)U_m(x) + f(x)\sum_{m=1}^{\infty} \alpha_m(m,t)U_m(x)$$
$$- f(x)\sum_{m=1}^{\infty} \beta_m(m,t)U_m(x) + \varepsilon_0\sum_{m=1}^{\infty} \dot{\alpha}_m(m,t)U_m(x)$$
$$- \varepsilon_0\sum_{m=1}^{\infty} \dot{\beta}_m(m,t)U_m(x) = R_1(x,t) \quad (15)$$

$$EJ\sum_{m=1}^{\infty} \beta_m(m,t)U_m^{iv}(x) + \mu\sum_{m=1}^{\infty} \ddot{\beta}_m(m,t)U_m(x) + f(x)\sum_{m=1}^{\infty} \beta_m(m,t)U_m(x)$$
$$- f(x)\sum_{m=1}^{\infty} \alpha_m(m,t)U_m(x) + \varepsilon_0\sum_{m=1}^{\infty} \dot{\beta}_m(m,t)U_m(x)$$

$$-\varepsilon_0 \sum_{m=1}^{\infty} \dot{\alpha}_m(m,t) U_m(x) = 0 \tag{16}$$

For the ease of simplification of equation (15) in particular, the dynamic load function $R_1(x,t)$ is further assumed as

$$R_1(x,t) = \sum_{m=1}^{\infty} \psi_m(m,t) U_m(x) \tag{17}$$

where $\psi_m(m;t)$ is the time function. Thus, for an arbitrary subscript k, equation (13) is substituted into equation (3), to obtain,

$$R_1(x,t) = \left[M_L g - M_L \left(\sum_{K=1}^{\infty} \ddot{\alpha}_k(k,t) U_k(x) + 2v \sum_{k=1}^{\infty} \dot{\alpha}_k(k,t) U_k'(x) + v^2 \sum_{k=1}^{\infty} \dot{\alpha}_k(k,t) U_k''(x) \right) \right] \delta(x - vt) \tag{18}$$

Now, a known normalized deflection function $V_j(x),\ j = 1, 2, 3, \ldots$ is applied on equation (17) initially. The orthonormality property for the normalized deflection curves $V_m(x),\ m = 1, 2, 3, \ldots$ is observed, and, the resulting equation from earlier operation performed is compared with equation (18), and an integral operation introduced along the beam's length thereby yielding the following time function.

$$\psi_m(m,t) = D_{11} + D_{12} + D_{13} + D_{14} \tag{19}$$

where

$$D_{11} = \frac{M_L g}{L} d_1(j) + 2\frac{M_L}{L} \sum_{n=1}^{\infty} \cos\frac{n\pi vt}{L} d_{11}(n,j) \qquad (19a)$$

$$D_{12} = -\frac{M_L}{L} \left[\sum_{k=1}^{\infty} \ddot{\alpha}_k(k,t)\, d_2 + 2\sum_{k=1}^{\infty} \ddot{\alpha}_k(k,t) \sum_{n=1}^{\infty} \cos\frac{n\pi vt}{L} d_{22}(j,k) \right] \qquad (19b)$$

$$D_{13} = -2\frac{M_L}{L} v \sum_{k=1}^{\infty} \dot{\alpha}_k(k,t) \left[d_3(j,k) + 2\sum_{n=1}^{\infty} \cos\frac{n\pi vt}{L} d_{33}(j,k) \right] \qquad (19c)$$

$$D_{14} = -\frac{M_L}{L} v^2 \sum_{k=1}^{\infty} \alpha_k(k,t) \left[d_4(j,k) + 2\sum_{n=1}^{\infty} \cos\frac{n\pi vt}{L} d_{44}(j,k) \right] \qquad (19d)$$

$$d_1(j) = \int_0^L V_j(x)dx\,; \quad d_{11}(j) = \int_0^L cos\frac{n\pi x}{L} \quad V_j(x)dx \qquad (19e)$$

$$d_2(j,k) = \int_0^L U_k(x)V_j(x)dx\,; \quad d_{22}(j,k) = \int_0^L cos\frac{n\pi x}{L} U_k(x)V_j(x)dx \qquad (19f)$$

$$d_3(j,k) = \int_0^L U_k'(x)V_j(x)dx\,; \quad d_{33}(j,k) = \int_0^L cos\frac{n\pi x}{L} U_k'(x)V_j(x)dx \qquad (19g)$$

$$d_4(j,k) = \int_0^L U_k''(x)V_j(x)dx\,; \quad d_{44}(j,k) = \int_0^L cos\frac{n\pi x}{L} U''_k(x)V_j(x)dx \qquad (19h)$$

In view of equations (19), (19a) – (19h), the obtained dynamic load function $R_1(x,t)$ is substituted into equation (15), to obtain,

$$EJ\sum_{m=1}^{\infty}\alpha_m(m,t)U_m^{iv}(x)+\mu\sum_{m=1}^{\infty}\ddot{\alpha}_m(m,t)U_m(x)+f(x)\sum_{m=1}^{\infty}\alpha_m(m,t)U_m(x)$$

$$-f(x)\sum_{m=1}^{\infty}\beta_m(m,t)U_m(x)+\varepsilon_0\sum_{m=1}^{\infty}\dot{\alpha}_m(m,t)U_m(x)-\varepsilon_0\sum_{m=1}^{\infty}\dot{\beta}_m(m,t)U_m(x)$$

$$=\sum_{m=1}^{\infty}\left[\frac{M_Lg}{L}\left[d_1(j)+2\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}d_{11}(n,j)\right]\right.$$

$$-\frac{M_L}{L}\sum_{k=1}^{\infty}\ddot{\alpha}_k(k,t)\left[d_2(j,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}d_{22}(j,k)\right]$$

$$-2\frac{M_L}{L}v\sum_{k=1}^{\infty}\dot{\alpha}_k(k,t)\left[d_3(j,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}d_{33}(j,k)\right]$$

$$\left.-\frac{M_L}{L}v^2\sum_{k=1}^{\infty}\alpha_k(k,t)\left[d_4(j,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}d_{44}(j,k)\right]\right]U_m(x) \quad (20)$$

For free vibration of a dynamic single Euler-Bernoulli beam system,

$$EJU_m^{iv}(x)-\mu\omega_m^2U_m(x)=0 \quad (21)$$

$$\omega_m^2=\frac{\lambda_m}{L^4}\frac{EJ}{\mu} \quad (22)$$

for which ω_m is the m^{th} natural circular frequency of the beam.

Equation (21) is substituted into equation (20) and some algebraic simplifications is performed on the obtained equation to yield,

$$\omega_m^2\alpha_m(m,t)+\ddot{\alpha}_m(m,t)+\frac{c}{\mu}[\alpha_m(m,t)-\beta_m(m,t)]$$

$$+k_3(x)[\alpha_m(m,t)-\beta_m(m,t)]+k_4(x)[\alpha_m(m,t)-\beta_m(m,t)]$$

$$+\frac{\varepsilon_0}{\mu}\dot{\alpha}_m(m,t)-\frac{\varepsilon_0}{\mu}\dot{\beta}_m(m,t)=\frac{M_L}{L}\left[d_1(j)+2\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}d_{11}(n,j)\right]$$

$$-\theta_1\sum_{k=1}^{\infty}\ddot{\alpha}_k(k,t)\left[d_2(j,k)+2\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}d_{22}(j,k)\right]$$

$$-2\theta_1v\sum_{k=1}^{\infty}\dot{\alpha}_k(k,t)\left[d_3(j,k)+2\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}d_{33}(j,k)\right]$$

$$-\theta_1v^2\sum_{k=1}^{\infty}\alpha_k(k,t)\left[d_4(j,k)+2\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}d_{44}(j,k)\right] \quad (23)$$

where,

$$k_3(x)=\frac{cc_1}{\mu}x;\quad k_4(x)=\frac{cc_2}{\mu}x^2;\quad \theta_1=\frac{M_L}{\mu L}\text{ (mass ratio)} \quad (23a)$$

Thus, the dynamic equation of motion which is meant for the upper beam have been simplified to yield a reduced coupled second order ordinary differential equation (23). A similar technique has been employed to simplify equation (16). Hence, the obtained equation is

$$\ddot{\beta}_m(m;t) + \omega_m^2\alpha(m;t) + \frac{c}{\mu}[\beta_m(m,t) - \alpha_m(m,t)]$$

$$+k_3(x)[\beta_m(m,t) - \alpha_m(m,t)] + k_4(x)\left[\dot{\beta}_m(m,t) - \alpha_m(m,t)\right]$$

$$+\frac{\varepsilon_0}{\mu}\dot{\beta}_m(m,t) - \frac{\varepsilon_0}{\mu}\dot{\alpha}_m(m,t) = 0 \qquad (24)$$

DECOUPLING OF EQUATIONS

At the moment, effort is geared toward obtaining a simplified form of the coupled second order ordinary differential equations. Following this, the decoupling is initially attained by considering a system of disjointed Euler-Bernoulli beams, with the corresponding end supports. In this case, it is assumed that the upper beam ($i = 1$) is acted upon by a concentrated moving mass while the lower beam ($i = 2$) vibrates freely. Hence, equations (23) and (24) reduces to,

$$\ddot{\alpha}_m(m;t) + \omega_m^2\alpha(m;t) + \theta_1\sum_{k=1}^{\infty}\ddot{\alpha}_k(k,t)\left[d_2(j,k) + 2\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}d_{22}(j,k)\right]$$

$$+2\theta_1 v\sum_{k=1}^{\infty}\dot{\alpha}_k(k,t)\left[d_3(j,k) + 2\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}d_{33}(j,k)\right]$$

$$+\theta_1 v^2\sum_{k=1}^{\infty}\alpha_k(k,t)\left[d_4(j,k) + 2\sum_{n=1}^{\infty}cos\frac{n\pi vt}{L}d_{44}(j,k)\right] = R_{0A}V_m(vt) \qquad (25)$$

$$R_{0A} = \theta_1 Lg \qquad (25a)$$

$$\ddot{\beta}_m(m;t) + \omega_m^2\,\beta_m(m,t) = 0 \qquad (26)$$

METHOD OF OBTAINING MODIFIED FREQUENCY

At the moment, it is observed that equations (25) and (26) are still coupled and difficult to solve. In order to overcome the solution difficulty, an approximate analytical scheme which is a modification of Struble's asymptotic technique [1,3] is introduced to simplify equation (25) in particular. This involves deriving a modified frequency due to the inertia effect of the mass of the moving load. To this end, each differential operator in equations (25) and (26) is replaced by an equivalent operator defined by the modified frequency. Hence, the mass ratio of the moving load to its length is represented by θ_1 and a small parameter θ is introduced such that,

$$\theta = \frac{\theta_1}{1+\theta_1} \qquad (27)$$

Effortlessly, one can verify that, $\theta_1 = \theta + O(\theta^2)$ (28)

To obtain the desired modified frequency, Struble's technique requires that the first approximate solution to the homogeneous part of equation (25) is presented as

$$\alpha_m(m.t) = \gamma(m.t)\cos[\omega_m t - \phi_m(m.t)] + \sum_{r=1}^{R}\lambda^r\alpha_r\,(m.t) + 0(\lambda^{R+1}) \qquad (29)$$

where $0 < R < \infty$, $\gamma(m,t)$ and $\phi_m(m,t)$ are slowly time varying functions of time wherein,

$$\dot{\phi}(m,t) \simeq O(\theta), \quad \ddot{\phi}(m,t) \simeq O(\theta^2) \qquad (30)$$

$$\dot{\gamma}(m,t) \simeq O(\theta), \quad \ddot{\gamma}(m,t) \simeq O(\theta^2) \qquad (31)$$

The notation " $\simeq$ " denotes "is of order"

By applying equation (29) on the homogeneous part of equation (25), and having taken equation (28) into consideration, the obtained equations after some algebraic simplification is,

$$
\begin{aligned}
&2\omega_m\theta(m;t)\dot{\alpha}(m;t)\cos[\omega_m t-\phi_m(m.t)] \\
&\quad -2\omega_m\theta(m;t)\sin[\omega_m t-\alpha_m(m.t)] \\
&\quad -2\lambda v\omega_m\theta(m;t)d_3(j.m)\sin[\omega_m t-\alpha_m(m.t)] \\
&\quad -4\lambda v\omega_m d_3(j.m)\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L}\sin[\omega_m t-\alpha_m(m.t)] \\
&\quad -\lambda\omega_m^2\theta(m;t)d_1(j.m)\cos[\omega_m t-\alpha_m(m.t)] \\
&\qquad -2\lambda\omega_m^2\theta(m;t)d_2(j.m)\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L}\cos[\omega_m t-\alpha_m(m.t)] \\
&\quad +\lambda v^2 d_3(j.m)\theta(m;t)\cos[\omega_m t-\alpha_m(m.t)] \\
&\qquad +2\lambda v^2 d_3(j.m)\theta(m;t)\sum_{n=1}^{\infty} cos\frac{n\pi vt}{L}\cos[\omega_m t-\alpha_m(m.t)]=0 \quad (32)
\end{aligned}
$$

In obtaining equation (32), every term in θ^2 and other terms in higher powers of θ have been considered nighgibly small. Thus, the corresponding variational equations as obtained from equation (32), have the form

$$-2\omega_m\dot{\gamma}(m.t)-2\lambda v d_3(j.m)\gamma(m.t)\omega_m=0 \quad (33)$$

$$2\omega_m\dot{\phi}_m(m.t)-\omega_n^2 2\lambda v d_1(j.k)+v^2 d_3(j.m)=0 \quad (34)$$

Solving equations (33) and (34), we have $\dot{\phi}(m,t)=P_0e^{-q_0t}$ (35) $\dot{\gamma}(m.t)=P_0e^{-q_0t}\cos(\rho_m t-I_m)$ (36)

where $P_0,\ q_0,\ I_m$ are constants, and,

$$\rho_m=\omega_m\left[1-\frac{\lambda}{2}\left(d_1(j,k)-\frac{v^2d_3(j,k)}{\omega_m^2}\right)\right] \quad (37)$$

is the particular modified frequency due to the presence of the effect of the moving mass of the load. Thus, the differential operator which acts on $\alpha_m(m.t)$ and $\alpha_k(k.t)$ on equation (25) as earlier mentioned, is thereby replaced by the equivalent free system operator defined by the modified frequency, ρ_m. As a result, equation (25) reduces to

$$\ddot{\alpha}_m(m,t)+\rho_m^2\alpha_m(m.t)=\theta_1 gLV_m(vt) \quad (38)$$

It is therefore, remarked that equations (38) and (26) respectively, are the simplified version of the original system for which interconnected nonlinear variable viscoelastic Winkler layer have been neglected. However, if the inner layer have been retained, the obtained transformed second order ordinary differential equations of the dynamic Euler-Bernoulli beams simplifies to,

$$
\begin{aligned}
&\ddot{\alpha}_m(m.t)+\rho_m^2\alpha_m(m,t)+\frac{c}{\mu}[\alpha_m(m,t)-\beta_m(m,t)] \\
&\qquad +k_3(x)[\alpha_m(m,t)-\beta_m(m,t)]+k_4(x)[\alpha_m(m,t)-\beta_m(m,t)] \\
&\quad +\frac{\varepsilon_0}{\mu}\left[\dot{\alpha}_m(m,t)-\dot{\beta}_m(m,t)\right]=\theta_1 gLV_m(vt) \quad (39)
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{\beta}_m(m.t)+\omega_m^2\beta_m(m,t)+\frac{c}{\mu}[\beta_m(m,t)-\alpha_m(m,t)] \\
&\qquad +k_3(x)[\beta_m(m,t)-\alpha_m(m,t)]+k_4(x)[\beta_m(m,t)-\alpha_m(m,t)]
\end{aligned}
$$

$$+\frac{\varepsilon_0}{\mu}\left[\dot{\beta}_m(m,t)-\dot{\alpha}_m(m,t)\right]=O \tag{40}$$

DIFFERENTIAL TRANSFORM METHOD

Based on the previous studies [], the iterative procedure of differential transform method have been deployed to simplify various differential equations due to vibration problems in engineering applications. Hence, the convenience of the method have been employed to solve the reduced transformed second order ordinary differential equations (39) and (40). The procedure involves considering the k^{th} derivative of a time function $\alpha_m(m,t)$ such that,

$$\bar{\alpha}_m(k) = \frac{1}{k!}\left[\frac{d^k \alpha_m(m,t)}{dt^k}\right]_{t=t_0} \tag{41}$$

The inverse differential transform of $\bar{\alpha}_m(k)$ is

$$\alpha_m(m,t) = \sum_{k=0}^{\infty} \bar{\alpha}_m(k)(t - t_0)^k \tag{42}$$

At $t_0 = 0$, equations (41) and (42) becomes

$$\alpha_m(m,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k \bar{\alpha}_m(m,t)}{dt^k}\right]_{t=t_0} \tag{43}$$

It has been established that, in practical applications, the result in equation (43) is finite, and it can be expressed as

$$\alpha_m(m,t) = \sum_{k=0}^{\mathrm{T}} \bar{\alpha}_m(k)\, t^k \tag{44}$$

such that $\sum_{k=\mathrm{T}+1}^{\infty} \bar{\alpha}_m(k) t^k$ is considered negligibly insignificant.

Table 1: Basic Theorem of Differential Transform Method for Equations of Motion

$u(t) + v(t)$	$\bar{U}(k) + \bar{V}(k)$
$cu(t)$	$c\bar{U}(k)$
$\frac{du(t)}{dt}$	$(k+1)\bar{U}(k+1)$
$\frac{d^n u(t)}{dt^n}$	$(k+1)(k+2) \ldots (k+n-1)(k+n)\bar{U}(k+n)$
t^m	$\delta(k-m) = \{\begin{smallmatrix}1, k=m\\0, k\neq m\end{smallmatrix}$
$sinat$	$\frac{1}{k!} a^k sin\left(\frac{k\pi}{2}\right)$
$cosat$	$\frac{1}{k!} a^k cos\left(\frac{k\pi}{2}\right)$
$sinhat$	$\frac{1}{2k!}[(a)^k - (-a)^k]$
$coshat$	$\frac{1}{2k!}[(a)^k + (-a)^k]$

Applying the Table 1 to equations (39) and (40), the obtained recurrence relations are

$$\bar{\alpha}_m(k+2) = \frac{1}{(k+1)(k+2)}\left[R_1 \frac{1}{k!}\left(\frac{m\pi v}{L}\right)^k sin\left(\frac{k\pi}{2}\right) - \rho_m^2 \bar{\alpha}_m(k) - Q_1 \bar{\alpha}_m(k) + Q_1 \bar{\beta}_m(k) - N_1(k+1)\bar{\alpha}_m(k+1) + N_1(k+1)\bar{\beta}_m(k+1)\right] \tag{45}$$

$$\bar{\beta}_m(k+2) = \frac{1}{(k+1)(k+2)}\left[Q_1 \bar{\alpha}_m(k) - \omega_m^2 \bar{\beta}_m(k) - Q_1 \bar{\alpha}_m(k) - N_1(k+1)\bar{\beta}_m(k+1) + N_1(k+1)\bar{\alpha}_m(k+1)\right] \tag{46}$$

Equations (45) and (46) have been solved in view of the following transformed initial conditions.

$$\bar{\alpha}_m(0) = 0 = \bar{\alpha}_m(1) \qquad (47)$$

$$\bar{\beta}_m(0) = 0 = \bar{\beta}_m(1) \qquad (48)$$

On substituting equations (47) and (48) appropriately for $k = 0, 1, 2, 3, ...$ into the recurrence relations (45) and (46), using "MAPLE 18", the obtained results are,

$$\bar{\alpha}_m(2) = 0 \qquad (49) \qquad \bar{\beta}_m(2) = 0 \qquad (50)$$

$$\bar{\alpha}_m(3) = \frac{R_1}{3!}\left(\frac{m\pi v}{L}\right) \qquad (51)$$

$$\bar{\beta}_m(3) = 0 \qquad (52)$$

$$\bar{\alpha}_m(4) = -\frac{N_1 R_1}{4!}\left(\frac{m\pi v}{L}\right) \qquad (53)$$

$$\bar{\beta}_m(4) = \frac{N_1 R_1}{4!} \qquad (54)$$

$$\bar{\alpha}_m(5) = \frac{R_1}{5!}\psi_m[2N_1^2 - (\psi_m^2 + \rho_m^2) - Q_1] \qquad (55)$$

$$\bar{\beta}_m(5) = \frac{R_1}{5!}\psi_m[Q_1 - 2N_1^2] \qquad (56)$$

$$\bar{\alpha}_m(6) = \frac{N_1 R_1}{6!}\psi_m[4(N_1^2 - Q_1) - \rho_m^2 - (\psi_m^2 + \rho_m^2)] \qquad (57)$$

$$\bar{\beta}_m(6) = \frac{N_1 R_1}{6!}\psi_m^2\left[4\left(N_1^2 - Q_1 - \omega_m^2 - (\psi_m^2 + \rho_m^2)\right)\right] \qquad (58)$$

Considering the results in equations (49) – (58) and in conjunction with the transformed initial conditions, the inverse differential transform in equation (44) is adequately applied. The resulting equatiottttns are,

$$W_1(x,t) = \sum_{m=1}^{\infty} \frac{R_1\psi_m}{(\psi_m^2 - \rho_m^2)}\left\{\frac{(\psi_m^2 - \rho_m^2)}{3!}t^3 - \frac{N_1(\psi_m^2 - \rho_m^2)}{4!}t^4 \right.$$
$$+ \frac{(\psi_m^2 - \rho_m^2)^2}{5!}\left[\frac{2N_1^2}{(\psi_m^2 - \rho_m^2)} - \frac{(\psi_m^2 + \rho_m^2)}{(\psi_m^2 - \rho_m^2)} - \frac{Q_1}{(\psi_m^2 - \rho_m^2)}\right]t^5 +$$
$$\left. - \frac{N_1(\psi_m^2 - \rho_m^2)^2}{6!}\left[\frac{4\left(N_1 - Q_1\right)}{(\psi_m^2 - \rho_m^2)} - \frac{\rho_m^2}{(\psi_m^2 - \rho_m^2)} - \frac{(\psi_m^2 + \rho_m^2)}{(\psi_m^2 - \rho_m^2)}\right]t^6 + \cdots\right\}\sin\frac{n\pi}{L}x \qquad (59)$$

$$W_2(x,t) = \sum_{m=1}^{\infty} \frac{R_1\psi_m}{(\psi_m^2 - \rho_m^2)}\left\{\frac{N_1(\psi_m^2 - \rho_m^2)}{4!}t^4 \right.$$
$$- \frac{(\psi_m^2 - \rho_m^2)^2}{5!}\left[\frac{Q_1}{(\psi_m^2 - \rho_m^2)} - \frac{2N_1^2}{(\psi_m^2 - \rho_m^2)}\right]t^5 - \frac{N_1(\psi_m^2 - \rho_m^2)^2}{6!}$$
$$\left. \times \left[\frac{4(N_1^2 - Q_1)}{(\psi_m^2 - \rho_m^2)} - \frac{\omega_m^2}{(\psi_m^2 - \rho_m^2)} - \frac{(\psi_m^2 + \rho_m^2)}{(\psi_m^2 - \rho_m^2)}\right]t^6 + \cdots\right\}\sin\frac{m\pi}{L}x \qquad (60)$$

Hence, equations (59) and (60) respectively, denote the dynamic lateral responses of the upper and lower simply supported Euler-Bernoulli beams with a non-linear variable viscoelastic interconnected layer of which is excited by a moving mass of constant magnitude.

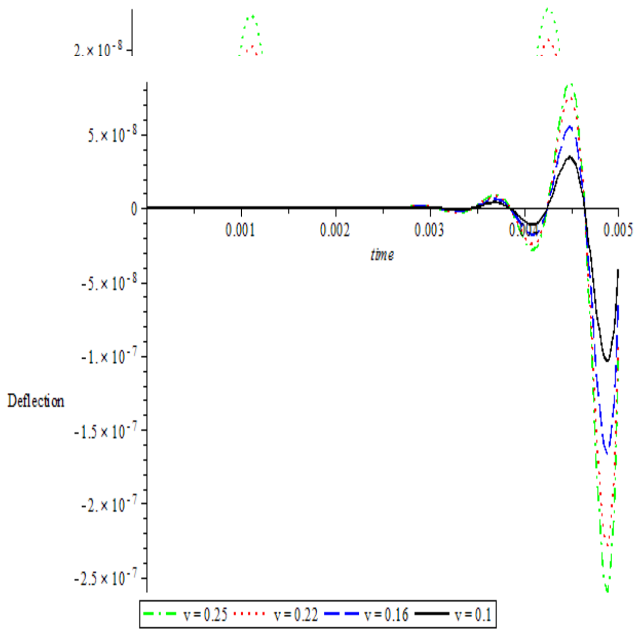

Figure 3(a): Graph of deflection against time at various velocities for the lower beam due to moving mass

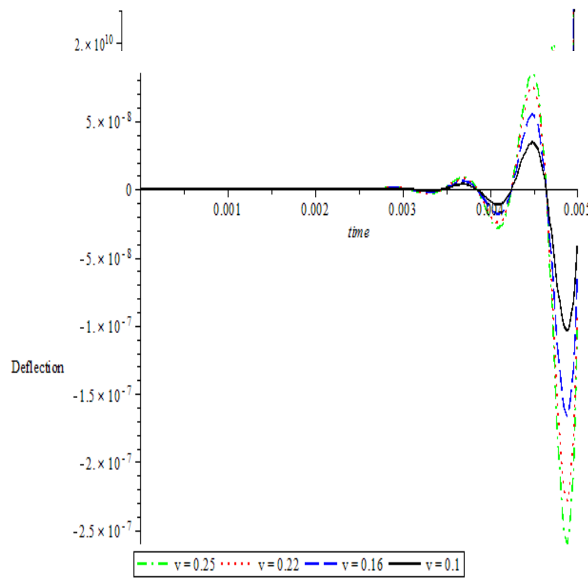

Figure 3(a): Graph of deflection against time at various velocities for the lower beam due to moving force

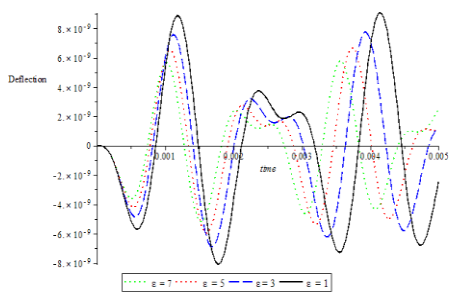

Figure 4(a): Graph of deflection against time varying the mass of the load for the upper beam

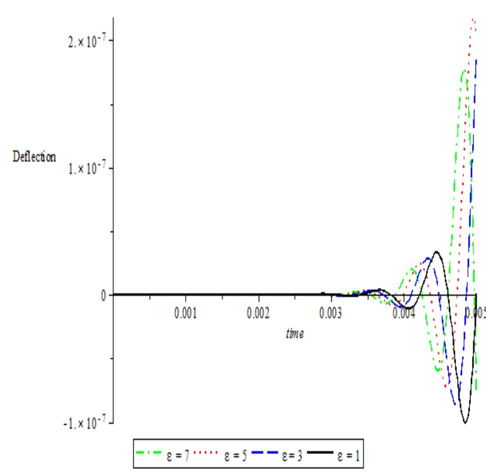

Figure 4(b): Graph of deflection against time

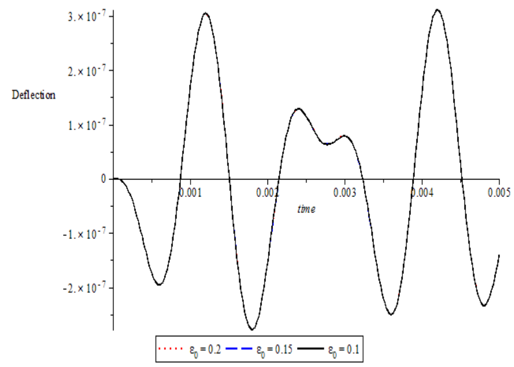

Figure 5(a): Graph of deflection against time varying the stiffness parameter at a constant velocity for the upper beam due to moving mass

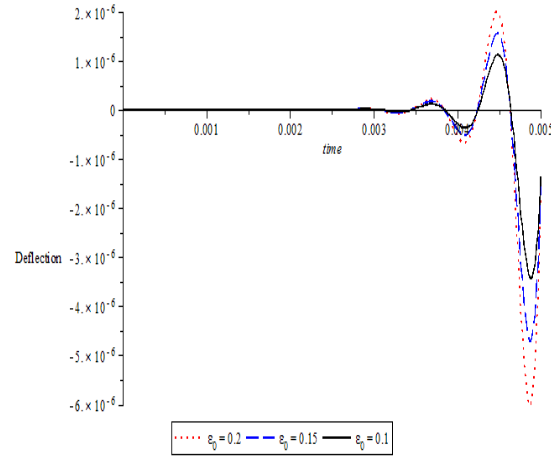

Figure 5(b): Graph of deflection against time varying the stiffness parameter at a constant

RESULT AND DISCUSSION

Without loss of generality, it is assumed that the two uniform elastically restrained end Euler-Bernoulli beams are geometrically and physically identical. Numerical results are presented in graphical form and discussed. The individual effects of various parameters such as load mass and velocity (v) of the concentrated moving load, elastic parameter (ε_0) and stiffness parameter of the interconnected layer $(k1)$ on the dynamic response of the simply supported double Euler-Bernoulli beam system traversed by a moving mass are examined and discussed. A computer program has been run for the following numerical data due to Mirzebeigy et al (2016). Figures 2(a) and 2(b) presents the influence of velocity on the traverse deflection of both upper and lower Euler-Bernoulli beams. The plots indicate that increase velocity to a decrease in the response amplitude is greater in the case involving lower beam due to moving force. Figure 3(a) and 3(b) represents the effect of variation of velocity on the traverse deflection of the upper Euler-Bernoulli due to moving mass and moving force respectively. It is noticeable that increasing the moving speed of the load caused an increase in the response amplitude of upper beam due to moving mass and force. The absolute maximum response amplitude is observed to be greater in the case due to moving force. Figure 4(a) and 4(b) presents the influence of mass ratio on the traverse deflection of both upper and lower Euler-Bernoulli beams. The plots indicate that increasing mass ratio to a decrease in the response amplitude of both upper and lower beams. However, the absolute maximum response amplitude is greater in the case involving lower beam due to moving force. In figures 5(a) and 5(b) depicts the effects of stiffness parameter on the dynamic response of both the upper and lower Euler-Bernoulli beam due to moving mass and moving force. It is observed that the dynamic response of the beam increases as the speed of the load increases.

CONCLUSION

The problem concerning the dynamic responses of two parallel elastic Euler-Bernoulli beams interconnected through a nonlinear variable viscoelastic Winkler-type layer under a moving concentrated load has been analyzed. The effect of the inertia of the moving load is the major concern of the study. The solution technique employed involve a variable separable method which has been used to reduce the fourth order partial differential equations characterizing the motion of the system to a set of coupled second order ordinary differential equations. The decoupling is attained by introducing an asymptotic method of Struble which has aided the simplification process thereby yielding a set of uncoupled second order ordinary differential equations of motion. The resulting equations is solved rising DTM.

Based on the analysis conducted so far, the effect of the various individual parameters such as constant load mass, velocity of the moving load, variable elastic parameter stiffness layer on the transverse deflections of both the upper and lower beams were examined. The following were the main findings:

Increasing the speed of the moving load is seen to cause an increase in the response amplitude of Euler-Bernoulli beam due to moving mass. The same trend is observed for the lower beam due to moving mass. Also, it is observed that an increase in the speed of the moving load has caused an increase in the response amplitude of deflection of both the upper and lower beams due to moving force.

Increase the mass ratio of both the upper and lower beams it is seen to cause a decrease in the response amplitude of deflections of the two beams. However, the amplitude of deflections due to the upper beam is higher while reverse is the case in the lower beam.

Increase the stiffness parameter of the linear interconnected layer led to an increase in the response amplitude of the deflections of the lower beam due to moving force. However, this variation of stiffness parameter, has produced the same response of deflection on the lower beam due to moving force.

Increasing the stiffness parameter of the linear interconnected layer is observed to cause an increase in the response amplitude of deflection of the lower beam due to moving mass. This variation of stiffness parameter, has produced the same response of deflection on the lower beam due to moving mass.

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