PROPOSAL OF NEW TWO-PARAMETER ESTIMATOR FOR GAMMA REGRESSION MODEL WITH CORRELATED REGRESSORS

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ABSTRACT

Multicollinearity among the explanatory variables in the gamma regression model makes the conventional maximum likelihood estimator (MLE) for estimating regression parameters in multiple regression analysis inefficient because its variance is high and unstable. Some researchers have presented estimators based on ridge and Liu biasing parameters to address the multicollinearity problem. In gamma regression models with collinearity among the explanatory variables, this work presents new two-parameter estimators. The conditions under which the suggested Gamma Modified New Two-Parameter (GMNTP) performs better are theoretically established, and simulation studies were also carried out. Both simulation and real-world application findings reveal that the GMNTP estimator with the K_2 shrinkage parameter outperforms the existing estimators in terms of MSE.

Keywords: *Gamma regression, Mean squared error, Multicollinearity, Shrinkage parameter, Two-parameter estimator.*

INTRODUCTION

In regression models, the distribution of the response variable is frequently a problem to solve. In real-world analyses, the response variable may not necessarily follow a normal distribution. For example, datasets obtained from social and economic databases typically contain positive values, resulting in positively skewed datasets. Similarly, epidemiological research is frequently positively skewed, which violates the normality of the response variable. When the dependent variable no longer follows a normal distribution but is positively skewed, the gamma regression model is applicable (Malehi, Pourmotahari, and Angali, 2015; Hattab, 2016; Amin, Qasim, and Amanullah, 2019; and Algamal, 2018a).

Maximum likelihood (ML) estimate, like Ordinary Least Squares (OLS), is the most commonly used approach for estimating Gamma regression model parameters under specified assumptions. When utilizing Gamma regression models, non-correlation assumptions must be considered. However, in many studies, the explanatory variables have a strong or near-strong linear relationship, resulting in the multicollinearity problem. When there is multicollinearity, the maximum likelihood (ML) coefficient estimate has a high variation, limiting its statistical significance (Kurtoglu, 2016; Amin, Qasim, Amanullah, and Afzal, 2017; Perez-Melo and Kibria, 2020). Furthermore, both inference and prediction might be affected by the sample variance of regression coefficients (Algamal, 2018a; Algamal and Asar, 2018; Algamal, 2018b; Algboory and Algamal, 2022).

There have been several proposals for dealing with multicollinearity in linear regression (LR) in the literature, including Stein estimator (Stein, 1956), principal component estimator (Massy, 1965), ridge regression estimator (Hoerl and Kennard, 1970), Liu estimator (Liu, 1993), Liu-type estimator (Liu, 2003), and modified Ridge-type estimator (Lukman, Ayinde, Binuomote, and Onate, 2019). A few of the proposed estimators in LR have been incorporated into gamma regression to mitigate multicollinearity. In 2016, Kurtoglu and Ozkale developed the Liu estimation, while Amin et al (2019) developed a gamma regression biased estimator with two parameters. But Lukman et al (2021), Kibria and Lukman's new ridge-type estimator was applied to the gamma regression model; Algboory and Algamal (2022) developed a Liu-Type estimator for gamma regression using (r-(k-d) class estimators and Idowu, Fasoranbaku and Ayinde (2023) proposed a two-parameter estimator for correlated regressors in gamma regression model.

This paper aims to generalize the modified new two-parameter (MNTP) estimator to the gamma regression model. The proposed estimator would supplement the alternative approach to mitigate the severe consequences of multicollinearity.

The paper is organized as follows: in Section 2, the Gamma Modified New Two-Parameter estimator is proposed and, then derived its mean squared error (MSE) properties. In section 3, a simulation study is conducted to evaluate the performance of the gamma-modified new two-parameter estimator and some existing estimators in the presence of multicollinearity. Also, real-life data is analyzed in section 4. Finally, some concluding remarks are provided in section 5.

Model and Estimators

The Gamma Regression Model is employed when the predictor variable is positively skewed and the mean is proportional to the dispersion parameter. (Qasim, Amin, & Amanullah, 2018; Amin et al., 2017). Consider a Gamma distribution with a nonnegative shape α and a nonnegative scale parameter τ, with the probability density function:

$$f(y_i) = \frac{1}{\Gamma(\alpha)\tau^{\alpha}} y_i^{\alpha-1} \lambda^{-\frac{y_i}{\tau}}, \quad y_i \geq 0 \tag{1}$$

where α is the non-negative shape parameter and τ is the scale parameter such that $E(y_i) = \mu_i = \alpha\tau = \theta_i$ which is also referred to as the canonical parameter and $Var(y_i) = \alpha\tau^2 = \frac{\theta_i^2}{\alpha}$, $\theta_i = \exp(x_i'\beta)$ where $x_i = (x_{i1}, x_{i2}, \ldots, x_{ip})'$, $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,p$ whereby n is the sample size and p is the number of explanatory variables such that (n > p). The log-likelihood function of eqn. (1) is given as:

$$l(\beta) = \sum_{i=1}^{n} \left[(\alpha-1)\ln(y_i) - \frac{y_i}{\tau} - \alpha\ln(\tau) - \ln(\Gamma(\alpha)) \right] \tag{2}$$

Since equation (2) are nonlinear in β it is solved iteratively using the Fisher scoring method as follows:

$$\beta^{(r+1)} = \beta^r + I^{-1}(\beta^r) S(\beta^r) \tag{3}$$

where r is the iteration degree, $I^{-1}(\beta) = \left[-E\left(\frac{\partial^2 l(\beta)}{\partial(\beta)\partial(\beta')} \right) \right]^{-1}$ and $S(\beta) = \partial l(\beta)/\partial\beta$. The process of iteration continues until the estimate converges. Hence, the ML estimator is obtained as:

$$\hat{\beta}_{MLE} = \left(X'\hat{W}X \right)^{-1} X'\hat{W}\hat{z} \tag{4}$$

where $\hat{W} = diag\left(\hat{\theta}_i^2\right)$ and $\hat{z}$ is the vector in ith element computed by, $\hat{z} = \hat{\theta}_i + \frac{y_i - \hat{\theta}_i}{\hat{\theta}_i^2}$ where $\hat{W}$ and $\hat{z}$ are computed at the final iteration by the procedure of Fisher scoring. (Hardin and Hilbe, 2012). The matrix of the mean squared error (MMSE) and the mean square error (MSE) MSE of the ML estimator is given respectively as follows:

$$MMSE\left(\hat{\beta}_{MLE}\right) = Cov\left(\hat{\beta}_{MLE}\right) = \phi\left(X'\hat{W}X\right)^{-1} \quad (5)$$

$$MSE\left(\hat{\beta}_{MLE}\right) = tr\left(MMSE\left(\hat{\beta}_{MLE}\right)\right) = \phi\sum_{j=1}^{p}\frac{1}{\lambda_j} \quad (6)$$

where $\phi = (n-p)^{-1}\sum_{i=1}^{n}\frac{(y_i - \theta_i)^2}{\theta_i^2}$.

In this case, λ_j is considered to be an eigenvalue of the matrix $A = X'\hat{W}X$. Decomposing the eigenvalues of the matrix A as follows: $A = q\Lambda q'$ for each eigenvalue in A, q is the orthogonal matrix consisting of their eigenvectors, given $A = diag(\lambda_1, \lambda_2, ..., \lambda_p)$.

When there is a linear correlation between the predictor variables, some of the eigenvalues will be small, indicating that the information matrix A is ill-conditioned. As a result, the MSE of the corresponding ML estimator will be overstated, and the signs of some regression coefficients may be misleading.

Gamma ridge estimator

The ridge estimator is a prominent estimator for reducing multicollinearity in linear regression models (Hoerl and Kennard, 1970). Also, Segerstedt (1992) introduced this notion into the Generalized Linear Model (GLM), and the gamma ridge estimator (GRE) is defined as:

$$\hat{\beta}_{GRE} = (A + kI)^{-1}A\hat{\beta}_{MLE} \quad (7)$$

where $A_k^{-1} = \left(X'\hat{W}X + kI\right)^{-1}$ and k > 0 is the biasing parameter. The MMSE and MSE of GRE are defined respectively as:

$$MMSE\left(\hat{\beta}_{GRE}\right) = Cov\left(\hat{\beta}_{GRE}\right) + Bias\left(\hat{\beta}_{GRE}\right)Bias\left(\hat{\beta}_{GRE}\right)'$$

$$= \phi A_k^{-1}AA_k^{-1} + k^2A_k^{-1}\beta\beta'A_k^{-1} \quad (8)$$

$$MSE\left(\hat{\beta}_{GRE}\right) = tr\left(MMSE\left(\hat{\beta}_{GRE}\right)\right)$$

$$= \phi\sum_{j=1}^{p}\frac{\lambda_j}{(\lambda_j + k)^2} + k^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j + k)^2} \quad (9)$$

where $\alpha = q'\beta$.

Gamma Liu estimator (GLE)

Liu (1993) presented the Liu estimator in linear regression models, which Kurtoglu and Ozkale (2016) applied to generalized linear models and named Gamma Liu estimator (GLE). The Gamma Liu estimator (GLE) is stated as follows:

$$\hat{\beta}_{GLE} = F_D\hat{\beta}_{MLE} \quad (10)$$

where $F_D = (A + I)^{-1}(A + dI)$, 0 < d < 1 is the Liu biasing parameter. The MMSE and MSE of GLE are defined respectively as:

$$MMSE\left(\hat{\beta}_{GLE}\right) = Cov\left(\hat{\beta}_{GLE}\right) + Bias\left(\hat{\beta}_{GLE}\right)Bias\left(\hat{\beta}_{GLE}\right)'$$

$$= \phi F_D A^{-1}F_D' + (1-d)^2(A + I)^{-1}\beta\beta'(A + I)^{-1} \quad (11)$$

$$MSE\left(\hat{\beta}_{GLE}\right) = tr\left(MMSE\left(\hat{\beta}_{GLE}\right)\right)$$

$$MSE\left(\hat{\beta}_{GLE}\right) = \phi\sum_{j=1}^{p}\frac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} + (1-d)^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j + 1)^2} \quad (12)$$

Gamma Liu-type estimator (GLTE)

Liu-type estimator proposed by Liu (2003) in linear regression model, adopted by Algamal and Asar (2018) in generalized linear models and termed Gamma Liu-type estimator (GLTE). The Gamma Liu-type estimator is expressed as:

$$\hat{\beta}_{GLTE} = A_k^{-1} A_D \hat{\beta}_{MLE} = F_{kD} \hat{\beta}_{MLE} \quad (13)$$

where $A_D = (A - dI)$, $F_{kD} = A_k^{-1} A_D$ and $-\infty < d < \infty$, k > 0. The MMSE and MSE of GLTE are defined respectively as:

$$MMSE(\hat{\beta}_{GLTE}) = Cov(\hat{\beta}_{GLTE}) + Bias(\hat{\beta}_{GLTE}) Bias(\hat{\beta}_{GLTE})'$$

$$= \phi F_{kD} A^{-1} F_{kD}' + (d+k)^2 A_k^{-1} \beta \beta' A_k^{-1} \quad (14)$$

$$MSE(\hat{\beta}_{GLTE}) = tr(MMSE(\hat{\beta}_{GLTE}))$$

$$MSE(\hat{\beta}_{GLTE}) = \phi \sum_{j=1}^{p} \frac{(\lambda_j - d)^2}{\lambda_j (\lambda_j + k)^2} + (d+k)^2 \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j + k)^2} \quad (15)$$

Proposed Gamma Ridge-Type Estimator

Ahmad and Aslam (2020) introduced the Modified New Two-Parameter (MNTP) estimator for the known linear regression model. The estimator has been defined as follows:

Let $\hat{\beta}^* = (S + kdI)^{-1} X' y$ and $S = X'X$, then

$$\hat{\beta}_{MNTP} = (S + I)^{-1} (S + dI) \hat{\beta}^* \quad (16)$$

In this study, the modified new two-parameter (MNTP) estimator is introduced into the gamma regression model and named Gamma Modified New Two-Parameter (GMNTP). The GMNTP is defined as:

$$\hat{\beta}_{GMNTP} = (A + I)^{-1} (A + dI) \hat{\beta}^*$$

$$\hat{\beta}_{GMNTP} = (A + I)^{-1} (A + dI)(A + kdI)^{-1} Z' y$$

$$\hat{\beta}_{GMNTP} = GA \hat{\beta}_{MLE} \quad (17)$$

where $G = (A + I)^{-1} (A + dI)(A + kdI)^{-1}$

We obtain the properties of Gamma Modified New Two-Parameter Estimator as:

$$Bias(\hat{\beta}_{GMNTP}) = E(\hat{\beta}_{GMNTP}) - \beta$$

$$Bias(\hat{\beta}_{GMNTP}) = [AG - I]\beta \quad (18)$$

$$= \sum_{i=1}^{p} \left[\frac{(1 - d + kd)\lambda_i + kd}{(\lambda_i + 1)^2 (\lambda_i + kd)^2} \right]^2 \alpha_i^2 \quad (19)$$

$$Cov(\hat{\beta}_{GMNTP}) = \phi GAG'$$

$$= \phi \sum_{i=1}^{p} \left[\frac{\lambda_i (\lambda_i + d)^2}{(\lambda_i + 1)^2 (\lambda_i + kd)^2} \right] \quad (20)$$

The MMSE and MSE in terms of eigenvalues are defined respectively as

$$MMSE(\hat{\beta}_{GMNTP}) = Cov(\hat{\beta}_{GMNTP}) + Bias(\hat{\beta}_{GMNTP}) Bias(\hat{\beta}_{GMNTP})'$$

$$= \phi GAG' + [AG - I]\alpha\alpha' [AG - I]' \quad (21)$$

$$MSE(\hat{\beta}_{GMNTP}) = tr(MMSE(\hat{\beta}_{GMNTP}))$$

$$MSE(\hat{\beta}_{GMNTP}) = \phi \sum_{i=1}^{p} \left[\frac{\lambda_i (\lambda_i + d)^2}{(\lambda_i + 1)^2 (\lambda_i + kd)^2} \right] + \sum_{i=1}^{p} \left[\frac{\{(1 - d + kd)\lambda_i + kd\}^2 \alpha_i^2}{(\lambda_i + 1)^2 (\lambda_i + kd)^2} \right] \quad (22)$$

Performance of Gamma Modified New Two-Parameter Estimators

We used Lemma 1 in this section to compare $\hat{\beta}_{GMNTP}$ with $\hat{\beta}_{MLE}$, $\hat{\beta}_{GRE}$, $\hat{\beta}_{GLE}$ and $\hat{\beta}_{GLTE}$ using smaller MSE criteria.

Lemma 1: Suppose R is an n × n matrix that is positive definite and α be a vector; then, $R-\alpha\alpha'$ is positive definite if and only if $\alpha' R^{-1}\alpha < 1$ (Farebrother, 1976).

As a result of this, we get the following result:

Lemma 2: Suppose that $\alpha_i = M_i y, i = 1,2$ be the two competing estimators of α. Assume that $I = Cov(\hat{\alpha}_1) - Cov(\hat{\alpha}_2) > 0$, then, $MMSE(\hat{\alpha}_1) - MMSE(\hat{\alpha}_2) > 0$ if and only if $v_2'(I + v_1 v_1') \le 1$, where v_i denotes the bias of $\hat{\alpha}_i$. (Trenkler and Toutenburg, 1990)

In the MSEM sense, the demonstration of $MMSE(\hat{\alpha}_1) - MMSE(\hat{\alpha}_2) > 0$ leads to the conclusion that $MMSE(\hat{\alpha}_2)$ is superior to $MMSE(\hat{\alpha}_1)$.

Comparison of GMNTP and MLE

Theorem 1: $\hat{\beta}_{GMNTP}$ is better than $\hat{\beta}_{MLE}$ if

$$\beta'(AG-I)'\left[\phi\left(A^{-1}-GAG'\right)\right]^{-1}(AG-I)\beta < 1 \quad (23)$$

Proof: The difference of the MSE is

$$V_1 = MSE(\hat{\beta}_{MLE}) - MSE(\hat{\beta}_{GMNTP})$$

$$= \sum_{j=1}^{p}\left\{\phi\left[\frac{1}{\lambda_i} - \frac{\lambda_i(\lambda_i+d)^2}{(\lambda_i+1)^2(\lambda_i+kd)^2}\right] - \left[\frac{[(1-d+kd)\lambda_i+kd]^2\alpha_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2}\right]\right\} \quad (24)$$

It can be seen from the above equation that V_1 is positive definite if

$$\phi[(1-d+kd)\lambda_i + kd] \ge \lambda_i\alpha_i^2[(1-d+kd)\lambda_i - kd]$$

. By Lemma 1, the proof is completed.

Comparison of GMNTP and GRE

Theorem 2: $\hat{\alpha}_{GMNTP}$ is better than $\hat{\alpha}_{GRE}$ if:

$$\beta'(AG-I)'\left[\phi\left(A_k^{-1}AA_k^{-1} - GAG'\right) + k^2A_k^{-1}\beta\beta'A_k^{-1}\right]^{-1}(AG-I)\beta < 1 \quad (25)$$

Proof: we investigate the difference between equation (22) and (9)

$$V_2 = MSE(\hat{\beta}_{GRE}) - MSE(\hat{\beta}_{GMNTP})$$

$$= \phi\sum_{i=1}^{p}\left[\frac{\lambda_i}{(\lambda_i+k)^2} - \frac{\lambda_i(\lambda_i+d)^2}{(\lambda_i+1)^2(\lambda_i+kd)^2}\right] + \sum_{i=1}^{p}\left[\frac{k^2}{(\lambda_i+k)^2} - \frac{[(1-d+kd)\lambda_i+kd]^2}{(\lambda_i+1)^2(\lambda_i+kd)^2}\right]\alpha_i^2 \quad (26)$$

$$V_2 = \frac{\phi\left[\lambda_i(\lambda_i+1)^2(\lambda_i+kd)^2 - \lambda_i(\lambda_i+d)^2(\lambda_i+k)^2\right] +}{(\lambda_i+1)^2(\lambda_i+kd)^2(\lambda_i+k)^2} \frac{\left[k^2\lambda_i(\lambda_i+1)^2(\lambda_i+kd)^2 - [(1-d+kd)\lambda_i+kd]^2(\lambda_i+k)^2\right]\alpha_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2(\lambda_i+k)^2} \quad (27)$$

It can be seen from the above equation that V_2 is positive definite if

$$\phi[\lambda_i(\lambda_i+1)(\lambda_i+kd) - \lambda_i(\lambda_i+d)(\lambda_i+k)] + [k\lambda_i(\lambda_i+1)(\lambda_i+kd) - [(1-d+kd)\lambda_i+kd](\lambda_i+k)]\alpha_i^2 > 0$$

. Hence, by lemma 1, the proof is completed.

Comparison of $\hat{\alpha}_{GMNTP}$ and $\hat{\alpha}_{GLE}$

Theorem 3: $\hat{\alpha}_{GMNTP}$ is better than $\hat{\alpha}_{GLE}$ if

$$\beta'(AG-I)'\left[\phi\left(F_DA^{-1}F_D' - GAG'\right) + (1-d)^2(A+I)^{-1}\beta\beta'(A+I)^{-1}\right](AG-I)\beta < 1 \quad (28)$$

Proof: we investigate the difference between equation (22) and (12)

$$V_3 = MSE(\hat{\beta}_{GLE}) - MSE(\hat{\beta}_{GMNTP})$$

$$V_3 = \phi \sum_{j=1}^{p} \left[\frac{(\lambda_i + d)^2}{\lambda_i(\lambda_i+1)^2} - \frac{\lambda_i(\lambda_i+d)^2}{(\lambda_i+1)^2(\lambda_i+kd)^2} \right]$$

$$+ \sum_{j=1}^{p} \left[\frac{(1-d)^2}{(\lambda_i+1)^2} - \frac{[(1-d+kd)\lambda_i + kd]^2}{(\lambda_i+1)^2(\lambda_i+kd)^2} \right] \alpha_i^2 \quad (29)$$

It can be seen from the above equation that V_3 is positive definite if $\left[\phi(\lambda_i kd + kd^2) + (d - \lambda_i d - kd^2 - \lambda\ kd)\alpha_i^2\right] > 0$. Hence, by lemma 1, the proof is completed.

Comparison of $\hat{\alpha}_{GMNTP}$ and $\hat{\alpha}_{GLTE}$

Theorem 4: $\hat{\alpha}_{GMNTP}$ is better than $\hat{\alpha}_{GLTE}$ if

$$\beta'(AG-I)'\left[\phi(F_{kD}A^{-1}F_{kD}' - GAG') + (d+k)^2 A_k^{-1}\beta\beta' A_k^{-1}\right] (AG-I)\beta < 1 \quad (30)$$

Proof: we investigate the difference between equation (22) and (15)

$$V_4 = MSE(\hat{\beta}_{GLTE}) - MSE(\hat{\beta}_{GMNTP})$$

$$V_4 = \phi \sum_{i=1}^{p} \left[\frac{(\lambda_i - d)^2}{(\lambda_i+k)^2} - \frac{\lambda_i(\lambda_i+d)^2}{(\lambda_i+1)^2(\lambda_i+kd)^2} \right]$$

$$+ \sum_{i=1}^{p} \left[\frac{(d+k)^2}{(\lambda_i+k)^2} - \frac{[(1-d+kd)\lambda_i - kd]^2}{(\lambda_i+1)^2(\lambda_i+kd)^2} \right] \alpha_i^2 \quad (31)$$

It can be seen from the above equation that V_4 is positive definite if

$$\phi\left[(\lambda_i - d)^2(\lambda_i+1)^2(\lambda_i+kd)^2 - \lambda_i(\lambda_i+d)^2(\lambda_i+k)^2\right]$$

$$\geq \left[(d+k)^2(\lambda_i+1)^2(\lambda_i+kd)^2 - (\lambda_i+k)^2\left[((1-d+kd)\lambda_i - kd)^2\right]\right]\alpha_i^2$$

Hence, by lemma 1, the proof is completed.

Selection of parameters k and d for Gamma Modified New Two-Parameter Estimator

For a fixed *d*, *k* optimal is obtained by differentiating equation (22) w.r.t. *k* and equating to 0, we have

$$k = \frac{\phi(\lambda_i + d) - (1-d)\lambda_i\alpha_i^2}{d(\lambda_i+1)\alpha_i^2} \quad (32)$$

However, *k* depends on the unknown ϕ and α_i. For practical purposes, they will be replaced by their unbiased estimator $\hat{\phi}$ and $\hat{\alpha}_i$. Hence, this will be obtained

$$\hat{k} = \frac{\hat{\phi}(\lambda_i + d) - (1-d)\lambda_i\hat{\alpha}_i^2}{d(\lambda_i+1)\hat{\alpha}_i^2} \quad (33)$$

Hoerl, Kennard, and Baldwin (1975) presented a harmonic mean for the biasing parameter k. Kibria (2003) also offered the geometric and arithmetic mean of the shrinkage parameter k values, which were proposed by Hoerl and Kennard (1970). As a result, four versions of shrinkage parameters are investigated for the proposed estimator Gamma Modified New Two-Parameter Estimator: arithmetic mean (AM), harmonic mean (HM), maximum (MX), and minimum (MN), which are defined as follows:

$$\hat{k}_{AM} = \frac{1}{p}\sum_{i=1}^{p} \frac{\hat{\phi}(\lambda_i + d) - (1-d)\lambda_i\hat{\alpha}_i^2}{d(\lambda_i+1)\hat{\alpha}_i^2} = k_1 \quad (34)$$

$$\hat{k}_{HM} = p\sum_{i=1}^{p} \frac{\hat{\phi}(\lambda_i + d) - (1-d)\lambda_i\hat{\alpha}_i^2}{d(\lambda_i+1)\hat{\alpha}_i^2} = k_2 \quad (35)$$

$$\hat{k}_{MX} = \max\left(\frac{\hat{\phi}(\lambda_i + d) - (1-d)\lambda_i\hat{\alpha}_i^2}{d(\lambda_i+1)\hat{\alpha}_i^2} \right) = k_3$$

(36)

$$\hat{k}_{MN} = \min\left(\frac{\hat{\phi}(\lambda_i + d) - (1-d)\lambda_i\hat{\alpha}_i^2}{d(\lambda_i + 1)\hat{\alpha}_i^2}\right) = k_4$$

(37)

The ideal *d* value can be thought of as the one that minimizes $MSE(\hat{\alpha}_{GMNTP})$. Then, by differentiating $MSE(\hat{\alpha}_{GMNTP})$ with respect to *d* and equating to 0, we get

$$d = \sum_{i=1}^{p}\left[\frac{\lambda_i(\alpha_i^2 - \phi)}{(\lambda_i\alpha_i^2 + \phi - k\lambda_i\alpha_i^2)}\right] \quad (38)$$

However, *d* depends on the unknown ϕ and α_i. For practical purposes, they will be replaced by their unbiased estimator $\hat{\phi}$ and $\hat{\alpha}_i$. Hence, this will be obtained

$$\hat{d} = \sum_{i=1}^{p}\left[\frac{\lambda_i(\hat{\alpha}_i^2 - \hat{\phi})}{(\lambda_i\hat{\alpha}_i^2 + \hat{\phi} - \hat{k}\lambda_i\hat{\alpha}_i^2)}\right] \quad (39)$$

Monte Carlo simulation

Monte Carlo simulation experiments were used to evaluate GMNTP performance with varying degrees of multicollinearity.

Simulation Design

The dependent variable for Gamma Regression model is generated as $y_i \sim Gamma(\mu_i, \phi)$, where $\phi \in \{0.5, 1, 1.5\}$ and $\mu = \theta_i = \exp(x_i'\beta)$, $\beta = (\beta_1, \beta_{2,}, ..., \beta_p)$ with parameter vector, β chosen such that $\sum_{j=1}^{p}\beta_j^2 = 1$ (Amin et al., 2017; Amin et al., 2019). Following (McDonald and Galarneau, 1975; Idowu et al., 2023; Owolabi et al., 2022 and Oladapo et al., 2022) the explanatory variables are obtained as follows:

$$x_{ij} = (1-\rho^2)^{1/2} w_{ij} + \rho w_{ip+1}, \quad i = 1,...,n \; j = 1,...,p \quad (40)$$

such that ρ is the correlation among the explanatory variables, w_{ij} are independent standard normal pseudorandom numbers. GMNTP estimator performance is considered under the following factors: explanatory variables (p = 4, 8), degree of correlation $(\rho = 0.8, 0.9, 0.95, 0.99)$, sample size (n = 30, 50, 100, 200, and 250). The average scalar MSE is determined using the R programming language with a repetition count of 1000.

$$SMSE(\hat{\beta}) = \frac{1}{1000}\sum_{i=1}^{1000}\sum_{j=1}^{p}(\hat{\beta}_{ij} - \beta_i)^2 \quad (41)$$

Simulation Result

Tables 1 and 2 show the MSE simulation results for GMNTP and various estimators under various conditions including sample size (n), degree of correlation (ρ), dispersion parameter (ϕ) and explanatory factors (p). The Gamma Modified New Two-Parameter (GMNTP) estimator with varying parameters consistently outperforms other existing estimators in terms of MSE. MLE is the least effective of these estimators. MSE increases when the number of explanatory variables (p) increases while the other components remain constant. Furthermore, as the value of the dispersion parameter and the level of correlation increases, so does the MSE. Also, increasing the sample size (n) reduces the MSE, particularly when rho equals 0.99. The proposed estimation method for shrinkage parameter k_2 is superior to others in almost all conditions.

Table 1: Estimated MSE of the estimators when p = 4

							GMNTP			
Phi	N	Rho	MLE	GRE	GLE	GLTE	k1	k2	k3	k4
0.5	30	0.8	0.0149	0.0065	0.0119	0.0089	0.0042	**0.0025**	0.0039	0.0089
		0.9	0.0252	0.0084	0.0180	0.0138	0.0045	**0.0027**	0.0037	0.0165
		0.95	0.0469	0.0123	0.0300	0.0249	0.0057	**0.0039**	0.0048	0.0325
		0.99	0.2326	0.0453	0.1269	0.1318	0.0150	**0.0052**	0.0076	0.1740
	50	0.8	0.6285	0.3492	0.5147	0.4276	0.2046	**0.1142**	0.1656	0.4464
		0.9	1.1167	0.5599	0.8446	0.7400	0.2882	**0.1248**	0.1981	0.7752
		0.95	2.1384	1.0050	1.5214	1.4474	0.4646	**0.1400**	0.2814	1.5470
		0.99	10.8000	4.8289	7.2241	8.9092	2.0245	**0.2143**	0.9893	9.3107
	100	0.8	0.7932	0.5207	2.4381	2.1131	0.3400	**0.2174**	0.3030	0.6007
		0.9	1.3796	0.8200	3.9337	3.6727	0.4530	**0.2392**	0.3532	1.0252
		0.95	2.6396	1.4758	7.0045	7.4492	0.6855	**0.2641**	0.4871	1.8792
		0.99	13.6849	7.3737	33.3629	52.6696	3.0749	**0.4358**	1.8007	10.5557
	200	0.8	4.3870	3.2085	4.1576	3.6406	2.2736	**1.5978**	2.1925	3.3637
		0.9	7.3852	4.8876	6.6596	6.0091	2.8798	**1.7318**	2.3073	5.5440
		0.95	13.9465	8.6621	11.8694	11.7308	4.0805	**1.8861**	2.8958	9.9555
		0.99	72.8142	43.6952	56.6148	79.0095	16.0639	**2.6223**	8.6068	55.2751
	250	0.8	4.1233	3.1365	3.9700	3.4428	2.3470	**1.6625**	2.3130	3.1883
		0.9	6.8783	4.7589	6.3689	5.5281	2.9927	**1.7917**	2.4541	5.2731
		0.95	12.9568	8.4418	11.4234	10.4484	4.2905	**1.9545**	3.1095	9.4952
		0.99	68.0184	42.9915	55.0690	63.3637	17.2005	**2.8318**	9.6763	51.2165
1	30	0.8	0.7801	0.3834	0.6166	0.4703	0.2130	**0.1334**	0.1743	0.4846
		0.9	1.3623	0.6061	0.9899	0.7884	0.2932	**0.1390**	0.2076	0.8534
		0.95	2.5376	1.0552	1.7211	1.4562	0.4587	**0.1445**	0.2815	1.6705
		0.99	12.0202	4.6744	7.5323	8.1191	1.9772	**0.1895**	0.9036	9.3553
	50	0.8	3.6309	1.8904	3.0223	2.4542	1.2716	**1.0358**	1.1955	2.3300
		0.9	5.9962	2.6577	4.4872	4.0361	1.4790	**1.0488**	1.2601	3.5965
		0.95	10.7846	4.2101	7.2632	7.7880	1.9503	**1.0697**	1.4641	6.4367
		0.99	49.5658	16.7653	28.8611	54.2878	6.6370	**1.2403**	3.6479	34.4537
	100	0.8	2.1565	1.3572	1.9903	1.6036	1.1097	**1.0178**	1.2402	1.3877
		0.9	3.1939	1.6626	2.7229	2.2386	1.2012	**1.0242**	1.1909	2.0018
		0.95	5.2938	2.2809	4.0474	3.6835	1.3851	**1.0338**	1.2089	3.1684
		0.99	22.2951	7.2855	13.6587	19.8254	3.2078	**1.0983**	2.0180	14.8362
	200	0.8	1.5598	1.1760	1.5140	1.2957	1.0516	**1.0084**	1.2364	1.0700
		0.9	2.0567	1.3242	1.9136	1.5999	1.0906	**1.0112**	1.2380	1.3328
		0.95	3.0602	1.6228	2.6324	2.2808	1.1684	**1.0157**	1.1794	1.9359
		0.99	11.1574	4.0255	7.4353	9.5689	1.8703	**1.0454**	1.4206	6.7711
	250	0.8	1.3970	1.1147	1.3706	1.1954	1.0336	**1.0050**	1.1807	1.0225
		0.9	1.7510	1.2116	1.6655	1.3978	1.0592	**1.0074**	1.2446	1.1370
		0.95	2.4663	1.4071	2.1987	1.8616	1.1108	**1.0107**	1.2186	1.5266
		0.99	8.2450	2.9834	5.6559	7.1854	1.5942	**1.0272**	1.2840	4.8470
1.5	30	0.8	4.4797	2.0069	3.5530	2.7547	1.0523	**0.8205**	0.9080	2.7349
		0.9	8.3580	3.4650	6.0475	5.2764	1.4605	**0.7992**	1.0658	4.9517
		0.95	16.3893	6.5288	10.9466	11.1270	2.4214	**0.8072**	1.5487	9.8507
		0.99	83.6563	32.6923	50.8018	77.1163	12.6268	**1.2515**	6.4708	61.2397
	50	0.8	2.6169	1.2618	2.2312	1.6504	0.8467	**0.7934**	0.8378	1.6054
		0.9	4.6907	1.9614	3.6281	2.9112	1.0328	**0.7632**	0.8619	2.7948
		0.95	9.0409	3.4617	6.3380	5.9362	1.4334	**0.7428**	1.0304	5.4027
		0.99	46.1570	16.6387	28.3520	42.1574	5.8380	**0.8046**	2.9850	32.5898
	100	0.8	1.7641	1.0446	1.6493	1.3268	0.7712	**0.7807**	0.9278	1.0926
		0.9	3.0968	1.5968	2.7184	2.3322	0.9004	**0.7452**	0.8905	1.9561
		0.95	5.9598	2.8234	4.8150	4.8307	1.2008	**0.7173**	0.9184	3.7260
		0.99	31.1881	14.1001	21.3911	37.0813	4.4526	**0.8020**	2.3876	20.6636
	200	0.8	1.3357	0.9182	1.2989	1.0687	0.7402	**0.7478**	0.9972	0.7740
		0.9	2.2848	1.3806	2.1516	1.7594	0.8684	**0.7090**	1.0176	1.4230
		0.95	4.3736	2.4441	3.9228	3.4310	1.1662	**0.6815**	0.9255	3.0089
		0.99	23.3358	12.6117	18.3290	22.9493	3.8688	**0.7675**	2.2006	15.1797
	250	0.8	1.2770	0.9186	1.2529	1.0464	0.7408	**0.7261**	1.0088	0.7371
		0.9	2.1959	1.4109	2.1063	1.7485	0.8931	**0.6927**	1.1269	1.3195
		0.95	4.2336	2.5507	3.9183	3.4610	1.2492	**0.6663**	0.9840	3.0397
		0.99	22.9051	13.5341	19.0210	23.2697	4.3700	**0.7187**	2.4254	15.4291

Highlighted values show the smallest MSE among all the estimators under consideration.

As a result, GMNTP with k_2 is considered the best choice in almost all scenarios since it minimizes the predicted MSE of the GMNTP estimator.

Application

We illustrate the theoretical performance of the proposed estimator with a real-life data set.

Table 2: Estimated MSE of the estimators when p = 8

							GMNTP			
Phi	N	Rho	MLE	GRE	GLE	GLTE	k1	k2	k3	k4
0.5	30	0.8	0.0155	0.0045	0.0092	0.0072	0.0021	**0.0011**	0.0015	0.0084
		0.9	0.0299	0.0070	0.0159	0.0124	0.0023	**0.0011**	0.0017	0.0203
		0.95	0.0609	0.0125	0.0305	0.0241	0.0024	**0.0012**	0.0019	0.0491
		0.99	0.3316	0.0638	0.1630	0.1331	0.0028	**0.0016**	0.0025	0.3162
	50	0.8	0.0107	0.0048	0.0085	0.0060	0.0028	**0.0014**	0.0019	0.0073
		0.9	0.0191	0.0080	0.0141	0.0102	0.0047	**0.0015**	0.0024	0.0148
		0.95	0.0364	0.0147	0.0251	0.0187	0.0085	**0.0016**	0.0032	0.0312
		0.99	0.1777	0.0709	0.1128	0.0889	0.0400	**0.0017**	0.0098	0.1713
	100	0.8	0.1259	0.0630	0.1068	0.1002	0.0270	**0.0159**	0.0208	0.0963
		0.9	0.2370	0.1112	0.1858	0.2159	0.0324	**0.0169**	0.0221	0.1801
		0.95	0.4712	0.2157	0.3441	0.5201	0.0486	**0.0180**	0.0250	0.3019
	200	0.8	2.2204	1.3454	2.0502	1.7723	0.6809	**0.4428**	0.6479	1.7062
		0.9	4.0599	2.2527	3.5480	3.3543	0.7836	**0.4674**	0.6167	3.1091
		0.95	8.0388	4.2761	6.6153	7.1648	0.9625	**0.4991**	0.6734	5.7151
		0.99	43.2503	22.8616	32.5947	49.3979	2.6474	**0.5799**	1.1422	30.4925
	250	0.8	0.8599	0.5295	0.8023	0.6973	0.2750	**0.1899**	0.2853	0.6504
		0.9	1.5613	0.8707	1.3835	1.3243	0.3043	**0.2002**	0.2514	1.1766
		0.95	3.0875	1.6372	2.5799	2.8791	0.3635	**0.2124**	0.2635	2.2560
		0.99	16.6886	8.7388	12.7245	22.6684	0.8230	**0.2442**	0.3487	11.7215
1	30	0.8	0.7858	0.3060	0.5753	0.4948	0.0819	**0.0581**	0.0635	0.5091
		0.9	1.4376	0.5253	0.9683	0.9551	0.0951	**0.0588**	0.0666	0.9133
		0.95	2.7519	0.9680	1.7380	2.0489	0.1357	**0.0596**	0.0718	1.8641
		0.99	13.3373	4.5381	7.8519	17.3284	0.4732	**0.0625**	0.1148	10.6071
	50	0.8	0.0480	0.0213	0.0372	0.0350	0.0091	**0.0073**	0.0077	0.0318
		0.9	0.0857	0.0343	0.0590	0.0648	0.0116	**0.0074**	0.0080	0.0510
		0.95	0.1624	0.0607	0.1002	0.1277	0.0141	**0.0075**	0.0081	0.0896
		0.99	0.7865	0.2771	0.4201	0.6558	0.0497	**0.0076**	0.0215	0.4890
	100	0.8	1.7821	0.8849	1.5894	1.2465	0.5515	**0.4992**	0.6778	1.1071
		0.9	2.9299	1.2235	2.3897	2.0487	0.5914	**0.5009**	0.5722	1.8682
		0.95	5.2439	1.9066	3.8215	3.8880	0.6687	**0.5030**	0.5470	3.2745
		0.99	23.8703	7.4098	14.0239	24.3379	1.3516	**0.5078**	0.6910	15.5879
	200	0.8	2.2872	1.3854	2.1765	1.7493	1.0511	**1.0044**	1.5529	1.2311
		0.9	3.4338	1.7175	3.0882	2.5504	1.0834	**1.0062**	1.4363	2.0338
		0.95	5.7451	2.3867	4.7118	4.3910	1.1492	**1.0085**	1.1537	3.6678
		0.99	24.3467	7.7710	15.3026	24.7091	1.7124	**1.0140**	1.1539	14.8919
	250	0.8	2.0662	1.3119	1.9842	1.6271	1.0405	**1.0033**	1.5198	1.1200
		0.9	3.0196	1.5827	2.7588	2.3163	1.0690	**1.0045**	1.5065	1.6962
		0.95	4.9419	2.1287	4.1432	3.9413	1.1256	**1.0059**	1.2187	3.0999
		0.99	20.4183	6.5211	13.0346	23.7061	1.6084	**1.0090**	1.1198	12.5250
1.5	30	0.8	1.1848	0.4292	0.8826	0.7604	0.1375	**0.1135**	0.1110	0.7390
		0.9	2.1902	0.7434	1.4904	1.5138	0.1713	**0.1105**	0.1113	1.3818
		0.95	4.2294	1.3837	2.6720	3.2364	0.2325	**0.1072**	0.1257	2.6641
		0.99	20.8009	6.6243	12.0578	20.1670	0.9231	**0.0995**	0.2458	15.0066
	50	0.8	1.0846	0.4510	0.8975	0.7047	0.1914	**0.1806**	0.1872	0.6949
		0.9	1.9873	0.7566	1.4933	1.3355	0.2378	**0.1756**	0.1743	1.2001
		0.95	3.8402	1.3909	2.6156	2.7535	0.3163	**0.1701**	0.1777	2.3592
		0.99	19.1634	6.7215	11.3262	17.0177	0.9946	**0.1570**	0.2831	13.4224
	100	0.8	2.9180	1.4598	2.6839	2.1527	0.8312	**0.9246**	1.2545	1.7129
		0.9	5.2376	2.3185	4.5079	4.0826	0.8948	**0.8971**	0.9418	3.5426
		0.95	10.1143	4.1702	7.9811	8.8382	1.0334	**0.8661**	0.8283	6.7377
		0.99	51.8174	20.5562	34.2034	65.7825	2.4094	**0.7899**	1.0722	34.2036
	200	0.8	1.9143	1.1345	1.8483	1.5131	0.7807	**0.9197**	1.3259	1.0035
		0.9	3.3594	1.7357	3.1318	2.7101	0.8098	**0.8914**	1.1339	2.2510
		0.95	6.4890	3.0915	5.7378	5.6295	0.8773	**0.8571**	0.8418	4.8369
		0.99	34.3060	15.7720	26.2282	40.2877	1.7022	**0.7746**	0.8573	23.5545

250	0.8	1.8074	1.1108	1.7561	1.4544	0.7761	**0.9127**	1.3403	0.9237
	0.9	3.1492	1.6853	2.9698	2.5995	0.8057	**0.8833**	1.2078	2.0588
	0.95	6.0694	2.9864	5.4647	5.4375	0.8808	**0.8495**	0.8645	4.5808
	0.99	32.1998	15.2283	25.3313	40.1089	1.6737	**0.7661**	0.8895	21.6785

Highlighted values show the smallest MSE among all the estimators under consideration.

This data was originally analyzed by Chatterjee and Hadi (1988) and later used by (Kurtoglu and Ozkale, 2016; Amin *et al*. 2017; Qasim *et al*., 2018 and Lukman *et al*., 2019). The response variable (y) is nitrogen dioxide concentrations. There are four explanatory variables considered in this study. These include average wind speed (in miles per hour; X_1), maximum temperature (in ^{0}F; X_2), insolation (in langleys per day; X_3) and stability factor (in ^{0}F; X_4). The data were shown to follow gamma distribution (Chatterjee and Hadi, 1988; Kurtoglu and Ozkale, 2016; Amin et al., 2019). The explanatory variables are standardized because the units of the variables are not the same.

Table 3: Output of Regression Coefficients and the corresponding mean square error

	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	MSE
MLE	-0.0509928906	0.0190843001	0.0029049295	-0.0042283130	0.0013320191
GRE	0.0001052015	0.0008414052	0.0030282945	0.0001040772	0.0008875818
GLE	-0.0509529820	0.0190823643	0.0029036502	-0.0042184239	0.0013301959
GLTE	-0.0509843536	0.0190812523	0.0029049501	-0.0042275891	4.7153746984
k_1	-0.0363718824	0.0183252440	0.0024551131	-0.0007943305	0.0007553115
k_2	-0.0363718687	0.0183252434	0.0024551127	-0.0007943268	**0.0007553111**
k_3	-0.0363735559	0.0183253114	0.0024551718	-0.0007947921	0.0007553643
k_4	-0.0363718733	0.0183252436	0.0024551128	-0.0007943280	0.0007553112

Highlighted values show the smallest MSE among all the estimators under consideration.

CONCLUSION

The Gamma Modified New Two-Parameter (GMNTP) estimator is used to solve the multicollinearity problem and find the optimal parameters for the gamma regression model. Additionally, some approaches for determining the shrinkage parameters k are provided. Monte Carlo simulations and real datasets are utilized to assess the effectiveness of the suggested estimator. Monte Carlo simulation experiments have demonstrated that GMNTP estimators with k_2 shrinkage parameters outperform MLE, GRE, GLE, and GLTE estimators across all situations. GMNTP is also excellent in estimating nitrogen dioxide concentrations, with the lowest MSE compared to other estimators. Our Monte Carlo simulation and numerical example indicate that the shrinkage parameter in the GMNTP estimator is the best strategy for addressing multicollinearity in a Gamma regression model.

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